Special Properties of Binary Relations - Section 7.10

**Transitivity**
A binary relation R on domain D is said to be *transitive* if for all a,b,c in D if aRb and bRc then aRc.

e.g. If D={a,b,c,d,e} and R={(a,b),(a,c),(a,d),(b,c),(b,e),(c,e)} then R is transitive because
aRb and bRc and aRc, and aRb and bRe and aRe, and bRc and cRe and bRe
but S={(a,b),(b,d),(a,c),(c,b)} is not transitive because aSb and bSd but NOT aSd.

What this means in the graph of a relation is that if there is any path between two nodes then there is an arc between the two nodes.

In S there is a path a->b->d but no arc a->c.

Examples: The relation < on the integers Z is transitive

The relation <> is not transitive because (for just one example) 3<>4 and 4<>3 but 3<>3 is false!

The subset relation on sets is transitive.

Consider the set S of all strings of letters. The prefix relation P on S is xPy if x is a prefix of y. For example, the string x=abc is a prefix of the string y=abcde. Is P transitive?

**Reflexivity**
If, for a relation R on domain D, for every a in D aRa then R is said to be *reflexive*. Graphically this means that there is an arc from every node to itself.

The relation <= on Z is reflexive. Because every integer is <= itself. The subset relation is reflexive. The < relation is NOT reflexive.
Symmetry and Antisymmetry
A binary relation $R$ on domain $D$ is symmetric if for all $a, b$ in $D$ if $aRb$ then $bRa$. Graphically this means that whenever there is an arc from $a$ to $b$ there is another arc from $b$ to $a$.

If $R$ is a relation on domain $D$ then the inverse of $R$, written $R^{-1}$, is the relation defined $R^{-1} = \{(a, b) \mid (b, a) \in R\}$. Thus, a symmetric relation is its own inverse.

EXAMPLES: The relation $=$ on $\mathbb{Z}$ is symmetric. So is $\neq$. The “is a cousin of” relation on people is symmetric (if $a$ is a cousin of $b$ then $b$ is a cousin of $a$). Unfortunately the “loves” relation on people is not symmetric because sometimes $a$ loves $b$ but NOT $b$ loves $a$.

A relation $R$ is said to be antisymmetric if whenever $aRb$ and $bRa$ then $a=b$. Graphically this means that there is never both an arc from $a$ to $b$ and an arc from $b$ to $a$ unless $a$ and $b$ are the same node.

EXAMPLES: The relation $\leq$ on $\mathbb{Z}$ is antisymmetric. The relation “is a parent of” on people is antisymmetric because if $a$ is a parent of $b$ then NOT $b$ is a parent of $a$.

Partial Orders and Total Orders
A partial order on a domain $D$ is a transitive, antisymmetric binary relation. A partial order $O$ is a total order if for each pair $a, b$ in $D$, either $aOb$ or $bOa$.

EXAMPLES: The relations $\leq$ and $\geq$ on $\mathbb{Z}$ are total orders and, thus, partial orders. The relations $<$ and $>$ are partial but not total because for $n$ in $\mathbb{Z}$ neither $n<n$ nor $n>n$. The subset relation $\subseteq$ is a partial order but not total because neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$.

A total order defines a linear sequence of the elements. e.g. the relation $R$ below on $\{a_1, a_2, \ldots, a_5\}$.

If we state that $R$ is transitive then we can remove arcs that are implied by longer paths to get a reduced graph of $R$. 

Section 7.10
Question: Consider the set of nodes N in a singly linked list. Is the relation “points to” on N a total order? A partial order? (No. No.) How about the relation “can be reached from?” (Both total and partial).

The subset relation on the power set of \{a,b,c\}
\((\{a,b,c\}^2 = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\})

Drawn as a reduced graph.

![Graph of subset relation on \{a,b,c\}^2]

**Equivalence Relations**

An *equivalence relation* is a binary relation that is reflexive, symmetric and transitive.

Examples: \(\leq\) on \(\mathbb{Z}\) is not because it is not reflexive. \(=\) on \(\mathbb{Z}\) is an equivalence relation.

Let \(N\) be the set of NBA teams. Then the relation “is in the same division as’” is an equivalence relation. Why?

1. Reflexive - For any team \(t\), \(t\) is in the same division as \(t\)
2. Symmetric - For teams \(a\) and \(b\) if \(a\) is in the same division as \(b\) then \(b\) is in the same division as \(a\).
3. Transitive - If \(a\) is in the same division as \(b\) and \(b\) is in the same division as \(c\) then \(a\) is in the same division as \(c\).

Another example: Let \(P\) be the set of all people and let \(R\) be the relation “has common parents with” Then \(R\) is an equivalence relation. PROOF:

1. Reflexive - Surely for any person Joe, Joe has common parents with Joe.
2. Symmetric - If Joe has common parents with Jil then Jil has common parents with Joe.
3. Transitive - If Jo has common parent with Jil and Jil has common parents with Jack then Joe has common parents with Jack.

Question: Is the relation on \(P\) “is a sibling of” an equivalence relation? (Not reflexive)
**Equivalence Classes**

An equivalence relation partitions (cuts up into disjoint subsets) its domain into *equivalence classes*. For an equivalence relation $R$ on a domain $D$, for each $a$ in $D$, $\text{class}(a) = \{b \mid aRb\}$. Notice that for the set of NBA teams the equivalence classes for the relation “is in the same division as” are just the NBA divisions.

![Diagram of NBA divisions]

What is the set of equivalence classes for the relation on people “has common parents with?”

**Closures of Relations**

From a relation that does not have some property (e.g. reflexivity, transitivity), we can form the closure of the relation for the property by adding just enough pair so that the property is satisfied.

**EXAMPLES:** Let $D=\{a,b,c,d,e,f\}$ and let $R$ be the relation on $D$ $R=\{(a,d),(a,c),(a,f),((b,f),(d,e),(d,f}\}$

This relation is not a partial order because it is not transitive ($aRd$ and $dRe$ but NOT $aRe$). It is, however, antisymmetric. We could make it transitive by adding (if its not already there) $aRe$ whenever $aRb$ and $bRc$. The result would be the transitive closure of $R$. And, in this case, because $R$ is antisymmetric, the transitive closure of $R$ is a partial order.

![Diagram of transitive closure of R]

**An Algorithm for Transitive Closure**

Sets $A=\{a,e,i,o,u\}$, $B=\{1,2,3\}$ and $C=\{w,x,y,z\}$
Relation R: A->B  
<table>
<thead>
<tr>
<th>1 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>e</td>
</tr>
<tr>
<td>i</td>
</tr>
<tr>
<td>o</td>
</tr>
<tr>
<td>u</td>
</tr>
</tbody>
</table>

Relation S: B->C  
<table>
<thead>
<tr>
<th>w x y z</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>e</td>
</tr>
<tr>
<td>i</td>
</tr>
<tr>
<td>o</td>
</tr>
<tr>
<td>u</td>
</tr>
</tbody>
</table>

The composition of R and S, written RS, is a relation from A to C defined aRSc iff there is a b in B such that aRb and bRc. The characteristic array of RS can be computed by something like matrix multiplication of R’s matrix and S’s matrix. That is, for each element a in A and c in C

\[ RS[a,c] = \sum_{b \in B} R[a,b] \cdot S[b,c] \]

e.g. RS[a,w] = (R[a,1] and S[1,w]) or (R[a,2] and S[2,w]) or (R[a,3] and S[3,w])

\[ = (1 \text{ and } 1) \text{ or } (0 \text{ and } 0) \text{ or } (1 \text{ and } 0) = 1 \text{ or } 0 \text{ or } 0 = 1 \]

**Interpretation:** We can get from a to c (aRSc) if we can get from a to 1 and from 1 to c (aR1 and 1Sc) or we can get from a to 2 and from 2 to c (aR2 and 2Sc) or we can get from a to 3 and from 3 to c (aR3 and 3Sc).

**Relation RS**

<table>
<thead>
<tr>
<th>w x y z</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>e</td>
</tr>
<tr>
<td>i</td>
</tr>
<tr>
<td>o</td>
</tr>
<tr>
<td>u</td>
</tr>
</tbody>
</table>

If we have a relation R ON a domain D

\[
\begin{array}{c|cccc}
| & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 1 & 1 & 1 \\
3 & 0 & 0 & 0 & 0 \\
\end{array}
\]

and we compose the relation with itself, e.g. RR written R^2, then we get elements related that were separated by two arcs in R.

\[
\begin{array}{c|cccc}
| & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
2 & 0 & 1 & 1 & 1 \\
3 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then if we OR R2 with R we get everything that was originally related plus transitivity two arcs long.

\[
\begin{array}{c|cccc}
| & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 \\
2 & 0 & 1 & 1 & 1 \\
3 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Carrying this reasoning long enough we get the transitive closure of R which is R OR R^2 OR R^3 OR R^4 (because there are only 4 elements in D).