Input-to-State Stability Analysis on Particle Swarm Optimization

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ABSTRACT

This paper examines the dynamics of particle swarm optimization (PSO) by modeling PSO as a feedback cascade system and then applying input-to-state stability analysis. Using a feedback cascade system model we can include the effects of the global-best and personal-best values more directly in the model of the dynamics. Thus in contrast to previous study of PSO dynamics, the input-to-state stability property used here allows for the analysis of PSO both before and at stagnation. In addition, the use of input-to-state stability allows this analysis to preserve random terms which were heretofore simplified to constants. This analysis is important because it can inform the setting of PSO parameters and better characterize the nature of PSO as a dynamic system. This work also illuminates the way in which the personal-best and the global-best updates influence the bound on the particle's position and hence, how the algorithm exploits and explores the fitness landscape as a function of the personal best and global best.

Categories and Subject Descriptors

I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search—*Heuristic methods*

Keywords

Particle Swarm Optimization; Input-to-State Stability

1. INTRODUCTION

Particle Swarm Optimization (PSO) is a popular and well-studied algorithm that was originally motivated by the flocking behaviors of birds and insects. Soon after its first publication it was discovered that the structure of the PSO algorithm is amenable to formal analysis using dynamical systems theory (sometimes referred to as dynamic systems) [5]. The use of this theory has informed the setting of parameters [16, 8], led to the proposal of new variants of the algorithm [5], and allowed for the analysis of the behavior of the algorithm [15], especially the behavior at stagnation, that is, when the algorithm fails to find better solutions [5].

While the study of the algorithm at stagnation is important and a significant first step, it only answers questions about the behavior at the point that PSO has degenerated into random search. At that point the algorithm can be mimicked by simply sampling from the appropriate distribution [12]. In this paper we extend the limited work that has been done to understand the behavior *before* stagnation, that is, when the unique mechanisms of PSO are directing the behavior of the algorithm.

By using a *feedback cascade model* we are able to include both what we refer to as the *position update* which comes from the PSO equations, but also the *input update*, that is, the effect of the personal and global best. This paper does so in contrast to prior work which focuses on the position update. A cascade model also allows us to make fewer assumptions in mapping from PSO to a dynamical system model. Using this model we are able to derive the conditions under which the process is input-to-state stable [9], prove bounds on both the particle motion and the mean of particle motion. The input-to-state stable conditions and the bounds can inform parameter adjustments and other properties that can, in turn, control the extent to which the algorithm explores or exploits the fitness landscape. This is especially valuable in the context of the design of future PSO variants.

The body of this paper is organized as described here. In section 3, we model the PSO dynamics as a feedback cascade system, which enables the input-to-state stability analysis. The definition of input-to-state stability (ISS) and its meaning in the context of PSO are also given. Section 4 shows that for particular parameter values, the position-update component of a particle is input-to-state stable. Using the ISS property we then give the bound on particle motion. We also use the ISS property in the context of the analysis of the moments (the mean and higher moments) of particle motion. In section 5, we use the ISS property to help analyze the dynamics of the particle. Using the ISS property of the input-update component, we can analyze the dynamics of the particle before and at stagnation.

2. RELATED WORK

Although the input-to-state stable analysis given in this paper can be applied to many versions of PSO, for this work we use the formulas from Kennedy's most recent definition of PSO[2] for the constricted position-update rule. The constricted position-update rule is

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$$v_{ij}(k+1) = \chi[v_{ij}(k) + \phi^P u_{ij}^P(k)(x_{ij}^P(k) - x_{ij}(k)) + \phi^G u_{ij}^G(k)(x_{ij}^G(k) - x_{ij}(k))], \quad (1a)$$

$$x_{ij}(k+1) = x_{ij}(k) + v_{ij}(k+1). \quad (1b)$$

$x_{ij}(k)$ represents the position of particle i in dimension j at time k. Similarly, $v_{ij}(k)$ represents the velocity of particle i in dimension j also at time k. $x_{ij}^G(k)$ and $x_{ij}^P(k)$ are global best (actually the topology best or local best) and personal best positions observed by the swarm and the particle respectively. $u_{ij}^G(k)$ and $u_{ij}^P(k)$ are independent random values drawn from $[0,1]$. $\chi \in (0,1)$, ϕ^P and ϕ^G are algorithm parameters.

Due to the stochastic nature of the particle's path and the social interaction represented by the topology, the dynamics of the algorithm is hard to evaluate in general. Understanding how the particles move guides how to improve the algorithm design [1], particularly the stochastic factors in the velocity update of Equation (1a). However once a particle is no longer able to find improvements in x^G and x^P, it exhibits the *stagnation phenomenon* [6]. In this state the analysis is easier since there is no effect from the topology.

Previous work that assumes stagnation can be categorized into two groups each based on how the analysis treats the stochastic factors. The first approach is to ignore the stochastic factors. Using this simplification, the convergence of a particle at stagnation can be analyzed [5, 3]. The convergence trajectories can be estimated [17] and the conclusions are compared with empirical results [4]. By building a linear system model [14], the PSO algorithm can be viewed as a closed loop system and the convergence can be analyzed. Based on such a convergence analysis, parameters can be set for best effect [16].

The second approach for handling the stochastic factors is based on stochastic analysis. By taking the mean of the stochastic variables, the stochastic terms can be converted into constant terms. A convergence analysis of the mean and variance of a particle at stagnation can also be obtained by using the characteristic equation in a discrete-time model [8]. In a similar way, other moments can be computed [12, 13, 11]. Using the discrete-time system model of different moments, the equilibrium can be found. The stability requirements can be obtained from the norm by setting the root values of the characteristic equation to all be less than 1.

There is also some work that addresses the dynamics when a particle is not in the stagnation phase. The discrete-time dynamics of PSO, that is, the dynamics of particle trajectory, can be approximated using a continuous-time model [7]. Furthermore, the probability of convergence in time can be analyzed by viewing the update process as a random search process [18]. The process of particles reaching a local optimum has also been analyzed [15].

3. INPUT-TO-STATE STABILITY OF PSO

Input-to-state stability analysis consists of two parts:

- the decomposition of the PSO algorithm into components, and
- the input-to-state stability of each component.

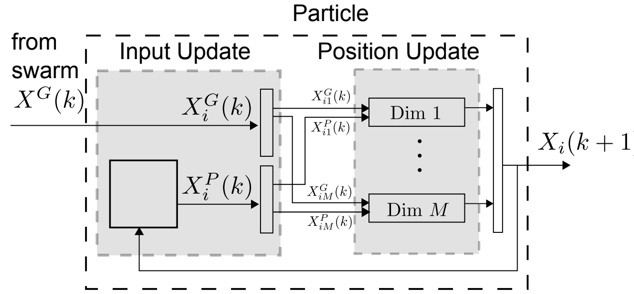

Figure 1: System structure of PSO

In this section, we model particle motion using a feedback cascade model. Then we review the definition of the input-to-state stability (ISS). We also explain why ISS should be applied to PSO.

3.1 Feedback Cascade Model

For the purpose of input-to-state stability analysis we decompose the PSO algorithm into components as shown in Figure 1. This decomposition is comprised of cascaded components (the input update, followed by the position update) and the feedback of the historical state. These two components are the *input-update component* for the global best ($x_i^G(k)$) and the personal best ($x_i^P(k)$), and the *position-update component* for particle position ($x_i(k+1)$), which depends on the inputs $x_i^G(k)$ and $x_i^P(k)$ as well as the previous position $x_i(k)$.

The properties of this system can be analyzed using the input-to-state stability of the position-update component and the input-update component. Given an input-to-state stable position-update component, we will see that the convergence of $x_i(k)$ depends on bounds on $x_i^G(k)$ and $x_i^P(k)$.

3.2 Input-to-state stability

Before reviewing the definition of input-to-state stability, we first introduce several types of functions [9].

- K-function $\mathbb{K}$: a function $\alpha : [0,a) \to [0,\infty)$ is continuous, strictly increasing and $\alpha(0) = 0$.
- K_∞-function $\mathbb{K}_\infty$: a function $\alpha : [0,a) \to [0,\infty)$ is a K-function and $\alpha(s) \to \infty$ as $s \to \infty$.
- KL-function $\mathbb{KL}$: a function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ satisfies:
  1. $\forall t \geq 0$, $\beta(\cdot, t)$ is a K-function;
  2. $\forall s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

These functions are used to define input-to-state stability in Definition 1.

Definition 1 (Input-to-state stable [9]). *For x, a discrete-time system defined as follows:*

$$x(k+1) = f(x(k), u(k)), \quad (2)$$

with $f(0,0) = 0$ [1]*, the system is* (globally) input-to-state stable *if there exist a KL-function β and a K-function γ*

[1]This means that $x = 0$ is an equilibrium of the 0-input system.

such that, for each input $u \in l_\infty^m$ and each $\xi \in \mathbb{R}^n$, it holds that $\forall k \in \mathbb{Z}^+$,

$$|x(k,\xi,u)| \le \beta(|\xi|,k) + \gamma(\|u\|). \tag{3}$$

The $\beta()$ term in Equation (3) defines an initial bound with a decaying property. The $\gamma()$ term in Equation (3) defines a bound determined by the input. This means that the $\beta()$ term gradually decreases to zero and the position is bounded by a range determined by the bound on the input.

3.3 Importance of input-to-state stability

Under certain conditions the dynamics of complex systems can be understood by first decomposing the system into a set of individual input-to-state stable components. We will take this approach with PSO. Parallel combination of input-to-state stable components yields a combined structure that is also input-to-state stable [10]. In the case of PSO and as shown in Figure 1, if each component (representing a single dimension) is input-to-state stable, the position-update component which combines all the dimensions is also input-to-state stable. Thus we have Property 1.

PROPERTY 1. *The position-update component is input-to-state stable if the position update in each dimension is input-to-state stable.*

This simplifies the analysis of the system since it allows us to consider each dimension separately. The serial connection and the feedback connection also lead to some interesting property from input-to-state stability, which will be discussed in Section 5.

The input-to-state stability analysis also provides the tool for the analysis of the convergence of PSO and the analysis of bounds on particle motion. PSO is designed to strike an effective balance between *exploring* and *exploiting* a fitness landscape. A bound on a particle's state is an indicator of the nature of that balance. When this bound is large the particle is exploring. However, as a particle finishes exploring and reach stagnation, a particle's position should converge.

Input-to-state stability implies that the state of the system is bounded in a range determined by the bounds on the input. Before stagnation, when the personal best and global best values have not converged, we can expect only a loose bound on the particle state. These looser bounds reflect both what is know about the update process itself and what is know about the inputs to the update process, that is, the personal best and the global best.

We call the bounds on the global best and personal best the "exploit radius" and the bounds on the particle's position a "explore radius". The ratio of the explore radius to the exploit radius is determined by the parameters of the position-update component. However, if the personal best and global best converge to an estimated optimal position, the exploit radius falls to zero and the explore radius converges to a bound.

4. ANALYSIS OF INPUT-TO-STATE STABILITY IN PSO

We then show PSO satisfies the definition of input-to-state stability when the parameters of PSO are set in a requisite range. We also derive the bounds implied by the ISS property and use the ISS property in Section 5 to find bounds

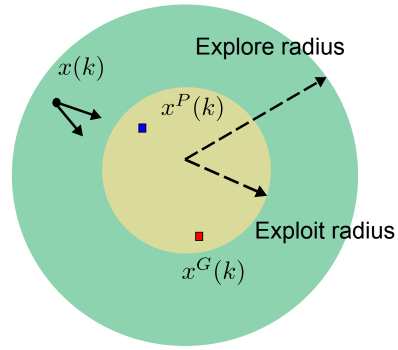

Figure 2: Exploration and exploitation.

on particle motion. Last, we will use ISS to analyze the moments (the mean and higher moments) of particle motion.

In our analysis of the PSO algorithm, we seek to understand how the particles converge to some position x^*, which is intended (not guaranteed) by the algorithm to be the global minimum position of the objective function.

For this analysis we use a one-dimension particle and extract the linear form of the position-update component. As noted above, the one dimensional case can be extended to many dimensions.

We begin our analysis of PSO input to state stability by rewriting the PSO equations in (1) in the following way:

$$X(k+1) = A(k)X(k) + B(k)U(k) \tag{4}$$

with

$$A(k) = \begin{bmatrix} \chi & -\chi\phi^G u^G(k) - \chi\phi^P u^P(k) \\ \chi & 1 - \chi\phi^G u^G(k) - \chi\phi^P u^P(k) \end{bmatrix}$$

and

$$B(k) = \begin{bmatrix} \chi\phi^G u^G(k) & \chi\phi^P u^P(k) \\ \chi\phi^G u^G(k) & \chi\phi^P u^P(k) \end{bmatrix}.$$

The system state is $X(k) = [v(k),\ x(k) - x^*]^T$, and the system input is $U(k) = [x^G(k) - x^*,\ x^P(k) - x^*]^T$ [2]. The convergence of this model means that $v(k) \to 0$ and $x(k) \to x^*$.

4.1 Conditions for input-to-state stability for position update in PSO

Using the definition of the PSO position update as given in Equation (4), PSO can be shown to be input-to-state stable as defined in definition 1.

THEOREM 1. *When $|\lambda_{\max}(A(k))| < 1$, the position-update component of PSO (4) is input-to-state stable.*

The proof follows the ISS-Lyapunov-function approach. The ISS-Lyapunov function, defined in Appendix 6.1, can be used to prove the input-to-state stability of a system and analyze the state bound[9]. The details of the proof are given in Appendix 6.2.

Note that in Equation (4), $[v(k),\ x(k) - x^*]^T = [0,\ 0]^T$ is an equilibrium position when the input $[x^G(k) - x^*,\ x^P(k) -$

[2]We use x^* to represent an equilibrium point to the system. In PSO, it can be a local optimum, a global optimum, or an estimated optimum. We use it as a reference point to check the bounds.

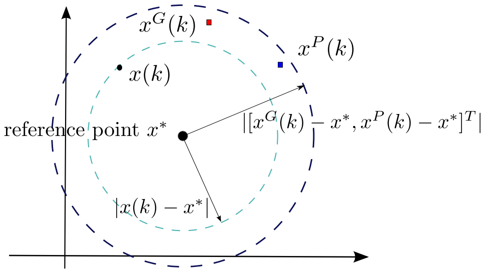

Figure 3: A bound on a particle's position by a reference point x^* from Equation (6). The ratio of two radii indicates γ.

$x^*]^T = [0,\ 0]^T$. Without loss of generality, for an arbitrary optimization problem x^* would typically not be at the origin. In such a problem, input-to-state stability means that the boundaries of $|v(k)|$ and $|x(k) - x^*|$ would be transformed and thus determined by $|x^G(k) - x^*|$ and $|x^P(k) - x^*|$, but the properties of ISS apply independent of where the function is centered.

Having shown that PSO is input-to-state stable we can now state a bound on particle position.

Corollary 1. *Given a bound on the input $||u||$ in the position-update component, we have the bound on the particle position from Equation (4).*

$$\forall k, |x(k) - x^*| \leq \max\left(|x(0) - x^*|, \gamma(|\left[x^G(k) - x^*,\ x^P(k) - x^*\right]^T|)\right), \quad (5)$$

in which $\gamma = \alpha_3^{-1} \circ \sigma$. ($\alpha_3$ and σ are defined in Appendix 6.2.)

Proof. *This is obtained from Remark 3.7 in [9] and by choosing P be a symmetric identity matrix. Furthermore we drop the velocity part because $|x(k) - x^*| \leq |[v(k),\ x(k) - x^*]^T|$.* □

The max part is needed to account for the effect of the starting point, represented by the first parameter. Eventually the effect of the starting point no longer affects the system, formally:

$$\exists T, \forall k \geq T, |x(k) - x^*| \leq \gamma(|[x^G(k) - x^*,\ x^P(k) - x^*]^T|). \quad (6)$$

Figure 3 gives an example on how a particle's boundary is determined by the personal best and global best.

Corollary 2. *Let $A(k) = \begin{bmatrix} \chi & -\chi\phi \\ \chi & 1 - \chi\phi \end{bmatrix}$, in which $\phi \in [0, \phi^{sup}]$, $\phi^{sup} = \phi^P + \phi^G$ and $\chi \in (0, 1)$. When $\phi^{sup} \in \left(0, \frac{2(1+\chi)}{\chi}\right)$, the system (4) is input-to-state stable.*

The proof is given in Appendix 6.3.

Figure 4 shows the parameter space. The x-axis is $\phi^{sup} = \phi^P + \phi^G$ and the y-axis is χ. The stable region in dark blue is obtained from eigenvalue test on Theorem 1. The red boundary is obtained from Corollary 2. It indicates the equivalence of the results from Theorem 1 and Corollary 2.

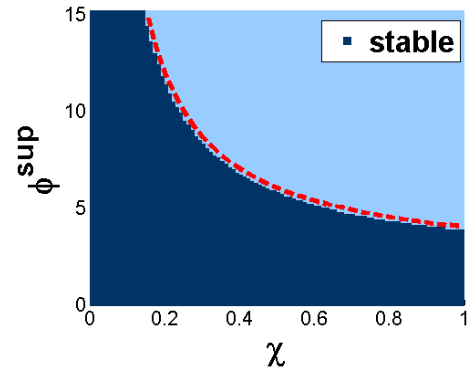

Figure 4: Parameter space

4.2 Moment Analysis

Using the same perspective of the feedback cascade system, the input-to-stable stability analysis can also be applied to moment analysis. Like the others [8, 11], we derive models for the statistical features (moments) of the particle's position at stagnation. In contrast to this prior work, input-to-state stability analysis can also provide bounds before a particle reaches stagnation.

We adopt the approach of Jiang, Luo & Yang [8] to construct an ISS model for the mean. Using that model $E(x(k))$ converges to

$$\hat{x} = \frac{\phi^P x^P + \phi^G x^G}{\phi^P + \phi^G}$$

in stagnation. If we treat $\hat{x}$ as a swarm average estimation on the optimum, we are interested in how $E(x(k))$ deviates from $\hat{x}$.

$$\begin{bmatrix} E(x(k+1)) - \hat{x} \\ E(x(k)) - \hat{x} \end{bmatrix} = A_m \begin{bmatrix} E(x(k)) - \hat{x} \\ E(x(k-1)) - \hat{x} \end{bmatrix} + B_m \begin{bmatrix} E(x^P(k)) - \hat{x} \\ E(x^G(k)) - \hat{x} \end{bmatrix}, \quad (7)$$

with

$$A_m = \begin{bmatrix} 1 + \chi - \frac{\chi\phi^P}{2} - \frac{\chi\phi^G}{2} & -\chi \\ 1 & 0 \end{bmatrix}$$

and

$$B_m = \begin{bmatrix} \frac{\chi\phi^P}{2} & \frac{\chi\phi^G}{2} \\ 0 & 0 \end{bmatrix}.$$

The convergence of $E(x(k))$ in stagnation is given in [8, 11]. Even without the stagnation assumption, the input-to-state stable analysis on Equation (7) indicates how $E(x(k))$ will deviate from $\hat{x}$ anytime we know how $E(x^G(k))$ and $E(x^P(k))$ deviate from $\hat{x}$. Stagnation is a simple case of knowing how $E(x^G(k))$ and $E(x^P(k))$ deviate from $\hat{x}$. This special case is discussed more later in this paper.

We now proceed to show the conditions that must hold for the mean model to be input-to-state stable.

Theorem 2. *The system (7) is input-to-state stable, if $|\lambda_{\max}(A_m)| < 1$.*

Proof. *The proof process is similar with Theorem 1, but we can get a constant symmetric positive definite Q_m from $A_m^T P A_m - P = -Q_m$.* □

Similar to Corollary 2, we have Corollary 3 for parameter selection on mean convergence. Note that when the condition in Corollary 2 is satisfied, the condition in Corollary 3 is also guaranteed. This means that when the system (4) is input-to-state stable, the mean dynamics (7) is also input-to-state stable.

COROLLARY 3. *Let* $A_m = \begin{bmatrix} 1+\chi-\frac{\chi\phi^P}{2}-\frac{\chi\phi^G}{2} & -\chi \\ 1 & 0 \end{bmatrix}$, *in which* $\phi \in [0, \phi^{sup}]$ *and* $\phi^{sup} = \phi^P + \phi^G$ *and* $\chi \in (0, 1)$. *When* $\phi^{sup} \in \left(0, \frac{4(1+\chi)}{\chi}\right)$, *the system* (7) *is input-to-state stable.*

The proof is given in Appendix 6.4.

Similar to Corollary 1, we can use the Q_m to determine the state bound.

COROLLARY 4. *If the system* (7) *is input-to-state stable, we have a bound*

$\exists T, \forall k > T,$

$$|E(x(k)) - \hat{x}| \leq \gamma_m | \left[E(x^P(k)) - \hat{x},\ E(x^G(k)) - \hat{x} \right]^T |, \tag{8}$$

with

$$\gamma_m = \frac{2\|A_m\|^2\|B_m\|^2 + \lambda_{min}(Q_m)^2\|B_m\|^2}{2(\lambda_{min}(Q_m))^3}. \tag{9}$$

In a similar way, we can apply the input-to-state stability analysis to the variance model [8] and higher order moment models [13].

5. IMPLICATIONS OF PARTICLE ISS

In this section, we add the input-update component that was first shown in Figure 1 and then analyze the convergence of particle position.

Since by Theorem 1 PSO is input-to-state stable, and therefore by Corollary 1 the stability of the cascade system depends on the output of the input-update component. We can say:

1. If the input-update component generates converging personal best and global best, the bound of the particle position will converge;
2. If the personal best and global best vary within a bound, the particle will converge within a bound;
3. If the personal best and global best become constant, the particle will converge within a bound.
4. If the personal best and global best are constant and the same, the particle will converge toward the global best.

By Theorem 2 and 4, we can make similar statements about the particle mean. As well, this boundary analysis could be applied to higher moments.

Furthermore, by Equation (5), we know that the convergence of a particle's position $x(k)$ to x^* depends on how $x^P(k)$ and $x^G(k)$ converge to x^* when the position-update component is input-to-state stable. In particular, the bound on the distance between a particle's position and x^* is determined by the initial distance $x(0) - x^*$, $x^P(k) - x^*$ and $x^G(k) - x^*$.

5.1 Stagnation

Since stagnation is defined as a state where a particle fails to find better positions, in stagnation $x^P(k)$ and $x^G(k)$ are constant in k, and can thus be represented as x^P and x^G respectively. If we assume that $u^P(k)$ and $u^G(k)$ are equivalent and constant, it can be stated that

$$\hat{x} = \frac{\phi^P x^P + \phi^G x^G}{\phi^P + \phi^G} \tag{10}$$

is an equilibrium point for stagnation as noted in previous work [5].

By Theorem 2, in stagnation the mean of the position will converge. Corollary 1 describes a bound on position as a function of the PSO parameters. Similarly, assuming that parameters are chosen that will also lead to the convergence of higher moments similar to previous work [8, 13], the pattern of particle movement at stagnation could be simulated by a distribution constructed to be consistent with PSO moment information[13].

By letting $x^* = \hat{x}$ be the reference point, and by Corollary 1, we can go beyond prior work and can identify a bound on PSO behavior at stagnation:

$$\exists T, \forall k > T, |x(k) - \hat{x}| \leq \gamma_d |[x^P - \hat{x},\ x^G - \hat{x}]^T|, \tag{11}$$

with

$$\gamma_d = \frac{2\|A'\|^2\|B'\|^2 + \lambda_{min}(Q'))^2\|B'\|^2}{2(\lambda_{min}(Q'))^3}. \tag{12}$$

Particularly, when $x^P = x^G$, we have $\hat{x} = x^G = x^P$. By Equation (11) we know that $\exists T, x(T) = x^G$, which means $x(k) \to x^G$. Thus we have shown the convergence of PSO in stagnation without treating the random terms as constants required by the work described in Section 2.

5.2 Before stagnation

Input-to-state stable analysis also supports understanding the cases before stagnation. When the $x^P(k)$ and $x^G(k)$ are not constant, the system state depends on the property of the input-update component. The personal-best update is

$$x_i^P(k) = \arg \max_{x \in \{x_i(k), x_i^P(k-1)\}} f(x). \tag{13}$$

The global-best update is

$$x_i^G(k) = \arg \max_{x \in \{x_i(k), x_i^G(k-1)\}} f(x). \tag{14}$$

As in Figure 1, there exists a feedback cascade system structure for a particle. If we assume that there is another model for swarm information sharing, which implements the $x^G(k)$ update. Then in the particle, the $x^G(k)$ can be viewed as an input that is independent with the current particle state. The feedback loop uses the current state to update the $x^P(k)$. As in Equation (13) and (14), the input-to-state stability of the input-update component depends on $f(x)$, which indicates that the input-to-state stability relies on the shape of the fitness distribution. Assuming that the PSO parameters are set such that the position-update component is input-to-state stable, there are three cases in analyzing the dynamics of a particle.

- *When only the* $x^G(k)$ *is constant*
  This happens usually when a "good" global best is found. Thus the swarm stops finding better global

bests. The input from the swarm can be modeled as a constant factor of the system. However, the particle still finds new personal best positions and updates the personal best. The system dynamics of the particle is determined by the feedback state, which updates the personal best. In this process, $f(x^P) < f(x^G)$, otherwise, x^G will be updated. In this case, the two components form a feedback loop structure. The small gain theorem [9] can be used. If the multiplication of the gain factors of two components is less than 1, the system will still converge.

- *When only the $x^P(k)$ is constant*
  This usually means that this particle is "stuck" in exploring a local region but some other particles are continuously finding new better position. Thus the global best is being updated. In this case, the feedback of the system does not impact the input-update component and thus nor does it impact the position-update component. In this case the system model can be simplified by ignoring the feedback. As a result, the system falls into only a serial cascade system because there is no feedback for this particle. If the input-update component is input-to-state stable, the serial connection of two input-to-state stable component is still input-to-state stable (it can be shown that a serial cascade of ISS components is also input-to-state stable [10]). Since this serial cascade system is input-to-state stable, the new position will be bounded somewhere around $x^G(k)$. This implies that the particle will converge toward the $x^G(k)$. If we have $f(x^G) > f(x^P)$, in the assumption of the continuity in the fitness space, when the particle gets closer to the global best, there exits some region that $f(x) > f(x^P)$. The x^P will start to change.

- *When both the $x^G(k)$ and $x^P(k)$ are not constant*
  This usually means that both the particle and the swarm it belongs to keep on finding better positions. Thus both the particle updates the personal best and the swarm updates the global best. In this case, the input-to-state stability of the input-update component is harder to guarantee. However, if the change of $x^G(k)$ and $x^P(k)$ is bounded, the movement of the particle is still bounded by Corollary 1.

Generally, in the PSO, the dynamics of a particle switches in between these cases before eventually reaching the stagnation. Understanding the dynamics of the particles before stagnation supports the exploration and exploitation capability of particles in optimal search.

6. CONCLUSION

In this paper, we have decomposed the particle in the PSO algorithm into a feedback cascade model, which consists of input-update and position-update components. We introduce the input-to-state stability analysis to the position-update component. For an input-to-state stable position-update component, if the input to this component is bounded, the state is bounded; also if the input to the component converges, the state converges. The convergence of a particle in PSO is determined by the output of the input-update component, which are the personal best and global best. If they are in stagnation, the particle converges.

The analysis of a cascade structure used here can be applied to a wide range of PSO variants. In the cases that the same position-update component but different input-update components are used, the convergence and the boundary of the particles are determined by whether the input-update component generates converging or bounded personal best and global best. For variants that use a different position-update component, the ISS properties would need to be verified.

The ISS property of the input-update component depends on the fitness distribution. We provide several scenarios for the dynamics of the particle. We show that the optimal search process switches among these scenarios and how the input-to-state stability analysis should be applied into different scenarios.

Appendix

6.1 ISS-Lyapunov function

Using the definitions of a K-function and a KL-function in Section 3 we can define an ISS-Lyapunov function as follows, an ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfies:

1. $\exists \alpha_1, \alpha_2 \in \mathbb{K}$ such that $\forall \xi \in \mathbb{R}^n$, $\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)$.
2. $\exists \alpha_3 \in \mathbb{K}_\infty, \sigma \in \mathbb{K}$ such that $\forall \xi \in \mathbb{R}^n, \forall \mu \in \mathbb{R}^m, V(f(\xi, \mu)) - V(\xi) \leq -\alpha_3(|\xi|) + \sigma(|\mu|)$.

6.2 Proof of Theorem 1

PROOF. Let P be an identity matrix. As $|\lambda_{\max}(A(k))| < 1$, we have $\|A^T(k)PA(k)\| \leq \|P\|\|A(k)\|^2 \leq \|P\||\lambda_{\max}(A(k))|^2 < \|P\|$. Because P is an identity matrix it is positive definite, and thus $A^T(k)PA(k)$ is positive definite or positive semi-definite by definition. So by positive definite ordering we have $A^T(k)PA(k) < P$.

Let $-Q(k) = A^T(k)PA(k) - P$. Since $A^T(k)PA(k) < P$ then $-Q(k) < 0$ furthermore $\exists Q' \forall k, Q(k) > Q' > 0$.

By the Lemma 3.5 in [9], if we can show that a proposed positive definite Lyapunov function is an ISS-Lyapunov function, the system is input-to-state stable.

Define a Lyapunov function

$$V(X(k)) = X^T(k)PX(k). \tag{15}$$

We can have $\lambda_{min}(P)|X(k)|^2 \leq V(X(k)) \leq \lambda_{max}(P)|X(k)|^2$ and $\lambda_{min}(P) = \lambda_{max}(P)$.

Let $\alpha_1(\xi) = \lambda_{min}\xi^2$ and $\alpha_2(\xi) = \lambda_{max}\xi^2$, we have $V(x)$ satisfying condition 1 of the ISS-Lyapunov function definition.

By applying Equation (4) to $V(X(k+1)) - V(X(k))$, we have

$$\begin{aligned}
&V(X(k+1)) - V(X(k)) \\
=&[X^T(k)A^T(k) + U^T(k)B^T(k)]P[A(k)X(k) + B(k)U(k)] \\
&- X^T(k)PX(k) \\
=&X^T(k)A^T(k)PA(k)X(k) + X^T(k)A^T(k)PB(k)U(k) \\
&+ U^T(k)B^T(k)PA(k)X(k) + U^T(k)B^T(k)PB(k)U(k) \\
&- X^T(k)PX(k)
\end{aligned} \tag{16}$$

As P is identity matrix, it is symmetric, thus

$$[X^T(k)A^T(k)PB(k)U(k)]^T = U^T(k)B^T(k)PA(k)X(k). \tag{17}$$

$V(X(k{+}1)), V(X(k)) \in \mathbb{R}$, we have $X^T(k)A^T(k)PB(k)U(k)$ and $U^T(k)B^T(k)PA(k)X(k)$ are both real value (like 1×1 matrix). Thus,

$$X^T(k)A^T(k)PB(k)U(k) = U^T(k)B^T(k)PA(k)X(k). \tag{18}$$

We then have

$$\begin{aligned} &V(X(k+1)) - V(X(k)) \\ &= - X^T(k)[A^T(k)PA(k) - P]X(k) \\ &\quad + U^T(k)B^T(k)PB(k)U(k) \\ &\quad + 2X^T(k)A^T(k)PB(k)U(k) \\ &\leq - X^T(k)Q'X(k) + U^T(k)B^T(k)PB(k)U(k) \\ &\quad + 2X^T(k)A^T(k)PB(k)U(k) \end{aligned} \tag{19}$$

By applying matrix norm, we have

$$\begin{aligned} &V(X(k+1)) - V(X(k)) \\ &\leq - \lambda_{min}(Q')|X(k)|^2 + |B^T(k)PB(k)||U(k)|^2 \\ &\quad + 2|A^T(k)PB(k)||U(k)||X(k)| \\ &= - \frac{1}{2}\lambda_{min}(Q')|X(k)|^2 + |B^T(k)PB(k)||U(k)|^2 \\ &\quad - \frac{1}{2}\lambda_{min}(Q')|X(k)|^2 + 2|A^T(k)PB(k)||U(k)||X(k)| \\ &= - \frac{1}{2}\lambda_{min}(Q')|X(k)|^2 \\ &\quad + \left(\frac{2|A^T(k)PB(k)|^2}{(\lambda_{min}(Q'))^2} + |B^T(k)PB(k)|\right)|U(k)|^2 \\ &\quad - \frac{1}{2}\lambda_{min}(Q')[|X(k)|^2 - \frac{4|A^T(k)PB(k)|}{\lambda_{min}(Q')}|X(k)||U(k)| \\ &\quad + \frac{4|A^T(k)PB(k)|^2}{(\lambda_{min}(Q'))^2}|U(k)|^2] \end{aligned} \tag{20}$$

By completing the square, we have

$$\begin{aligned} &V(X(k+1)) - V(X(k)) \\ &\leq - \frac{1}{2}\lambda_{min}(Q')|X(k)|^2 \\ &\quad + \left(\frac{2|A^T(k)PB(k)|^2}{(\lambda_{min}(Q'))^2} + |B^T(k)PB(k)|\right)|U(k)|^2 \\ &\quad - \frac{1}{2}\lambda_{min}(Q')\left(|X(k)| - \frac{2|A^T(k)PB(k)|}{\lambda_{min}(Q')}|U(k)|\right)^2 \\ &\leq - \frac{1}{2}\lambda_{min}(Q')|X(k)|^2 \\ &\quad + \left(\frac{2\|A^T(k)PB(k)\|^2}{(\lambda_{min}(Q'))^2} + \|B^T(k)PB(k)\|\right)|U(k)|^2. \end{aligned} \tag{21}$$

Because $u^P(k) \in [0, 1]$, there exist an A' and B' such that $\|A(k)\| \leq \|A'\|$ and $\|B(k)\| \leq \|B'\|$. We have $\|A^T(k)PB(k)\| \leq \|A'\|\|P\|\|B'\|$ and $\|B^T(k)PB(k)\| \leq \|P\|\|B'\|^2$.
Since the identity matrix P has $||P|| = 1$:

$$\begin{aligned} &V(X(k+1)) - V(X(k)) \\ &\leq - \frac{1}{2}\lambda_{min}(Q')|X(k)|^2 + \left(\frac{2\|A'\|^2\|B'\|^2}{(\lambda_{min}(Q'))^2} + \|B'\|^2\right)|U(k)|^2. \end{aligned} \tag{22}$$

Let

$$\alpha_3(\xi) = \frac{1}{2}\lambda_{min}(Q')\xi^2,$$

and

$$\sigma(\xi) = \left(\frac{2\|A'\|^2\|B'\|^2}{(\lambda_{min}(Q'))^2} + \|B'\|^2\right)\xi^2.$$

Thus we have $V(X(k+1)) - V(X(k))$ satisfying condition 2 of the ISS-Lyapunov function definition and so (15) is an ISS-Lyapunov function. Using Jiang's Lemma 3.5[9], the position-update component of PSO (Equation (4)) is input-to-state stable. □

6.3 Proof of Corollary 2

PROOF. Let $a = (1 + \chi) - \chi\phi$. The eigenvalues of $A(k)$ are

$$\lambda = \frac{a \pm \sqrt{a^2 - 4\chi}}{2}.$$

There can be two cases.

1. If $a^2 \geq 4\chi$, the eigenvalues are real number. We have $a \geq 2\sqrt{\chi}$ or $a \leq -2\sqrt{\chi}$.
   If $a \geq 2\sqrt{\chi}$, then $|\lambda_{\max}| < 1$ derives
   $$0 < \frac{a - \sqrt{a^2 - 4\chi}}{2} \leq \frac{a + \sqrt{a^2 - 4\chi}}{2} < 1.$$
   It means that $2\sqrt{\chi} \leq a < 1 + \chi$.
   If $a \leq 2\sqrt{\chi}$, then $|\lambda_{\max}| < 1$ derives
   $$-1 < \frac{a - \sqrt{a^2 - 4\chi}}{2} \leq \frac{a + \sqrt{a^2 - 4\chi}}{2} < 0.$$
   It means that $-(\chi + 1) < a \leq -2\sqrt{\chi}$.
2. If $a^2 < 4\chi$, the eigenvalues are complex number. We have $-2\sqrt{\chi} < a < 2\sqrt{\chi}$.
   $|\lambda_{\max}| < 1$ derives
   $$\frac{a^2}{4} + \frac{a^2 - 4\chi}{4} < 1.$$
   It means that $-2\sqrt{2(1 + \chi)} < a < 2\sqrt{2(1 + \chi)}$. Because $\sqrt{2(1 + \chi)} > 2\sqrt{\chi}$, we have $-2\sqrt{\chi} < a < 2\sqrt{\chi}$.

Combining these two cases, we have $-(1 + \chi) < a < 1 + \chi$. It equals to $\phi \in \left(0, \frac{2(1+\chi)}{\chi}\right)$. □

6.4 Proof of Corollary 3

PROOF. The proof is similar with that in Subsection 6.3. In this case, $a = (1 + \chi) - \frac{\phi}{2}\chi$. Similarly, we can have two cases and derive $-(1 + \chi) < a < 1 + \chi$. It equals to $\phi \in \left(0, \frac{4(1+\chi)}{\chi}\right)$. □

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