Chapter 8

Turing Machine (TMs)

Turing Machines (TMs)

- Accepts the languages that can be generated by *unrestricted* (*phrase-structured*) grammars

- No computational machine (i.e., computational language recognition system) is *more powerful* than the class of TMs due to the language processing power, i.e., the *generative power of grammars*, its unlimited memory, and *time of computations*

- Proposed by Alan Turing in 1936 as a result of *studying algorithmic processes* by means of a *computational model*

- TMs are similar to FSAs since they both consist of

  i) a *control mechanism*, and
  ii) an *input tape*

  In addition, TMs can

  i) *move* their tape head *back* and *forth*, and
  ii) *write* on, as well as *read* from, their tapes.

Turing Machines

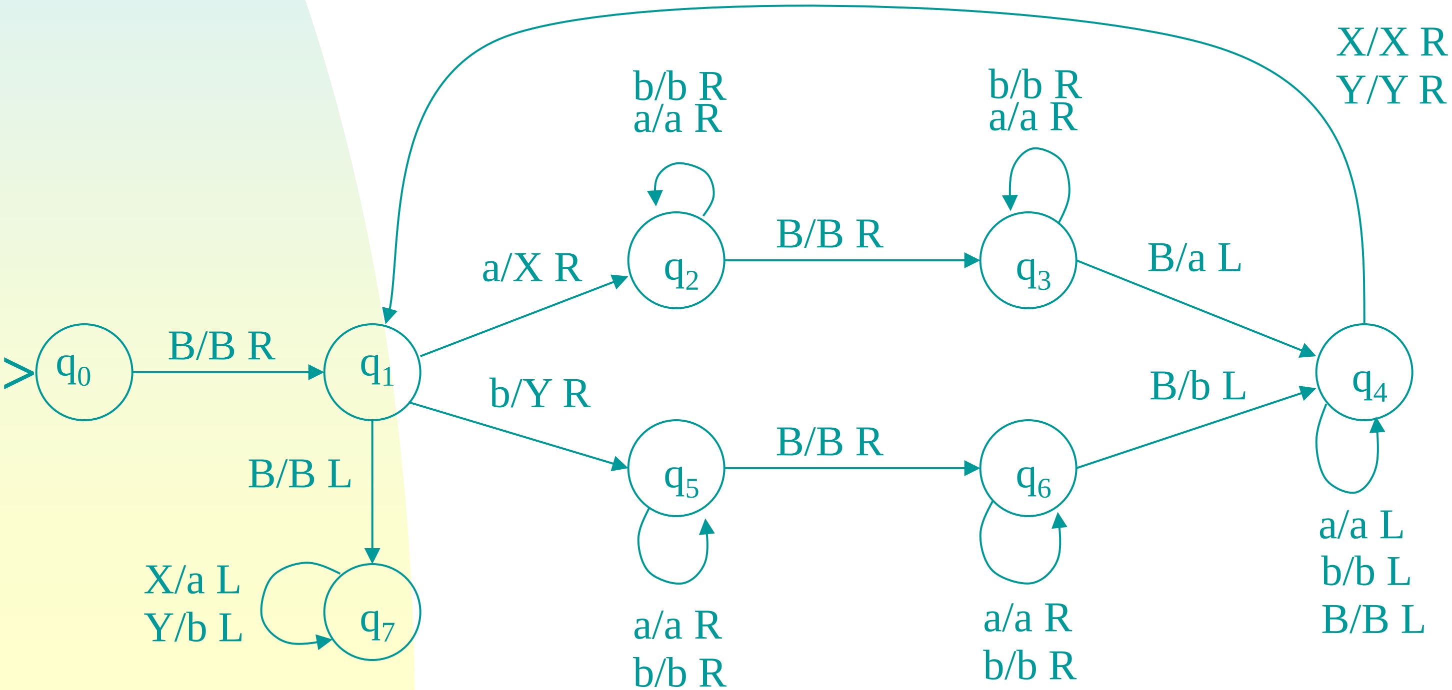
- Defn. 8.1.1 A TM is a quintuple $M = (Q, \Sigma, \Gamma, \delta, q_0)$, where
  - Q is a finite set of states
  - Γ is a finite set called the tape alphabet which contains B, a special symbol that represents a *blank*
  - $\Sigma \subseteq \Gamma - \{ B \}$, is the input alphabet
  - $\delta: Q \times \Gamma \to Q \times \Gamma \times \{ L, R \}$, a transition function, which is a *partial* function
  - $q_0 \in Q$, is the start state

Turing Machines

- Machine Operations:
  - Write operation - replaces a symbol on the tape with another (not necessarily distant) symbol
  - Move operation - moves the tape head one cell to the right (left, respectively) and then *shift* to a new (or current) state
  - Halt operation - halts when the TM encounters a < *state*, *input symbol* > pair for which no transition is defined

Turing Machines

- Machine Operations. TMs are designed to perform *computations* on strings from the input alphabet
- Example 8.1.2. A TM produces a *copy* of input string over { *a*, *b* } with input *BuB* and terminates with tape *BuBuB*, where $u \in (a \cup b)^*$



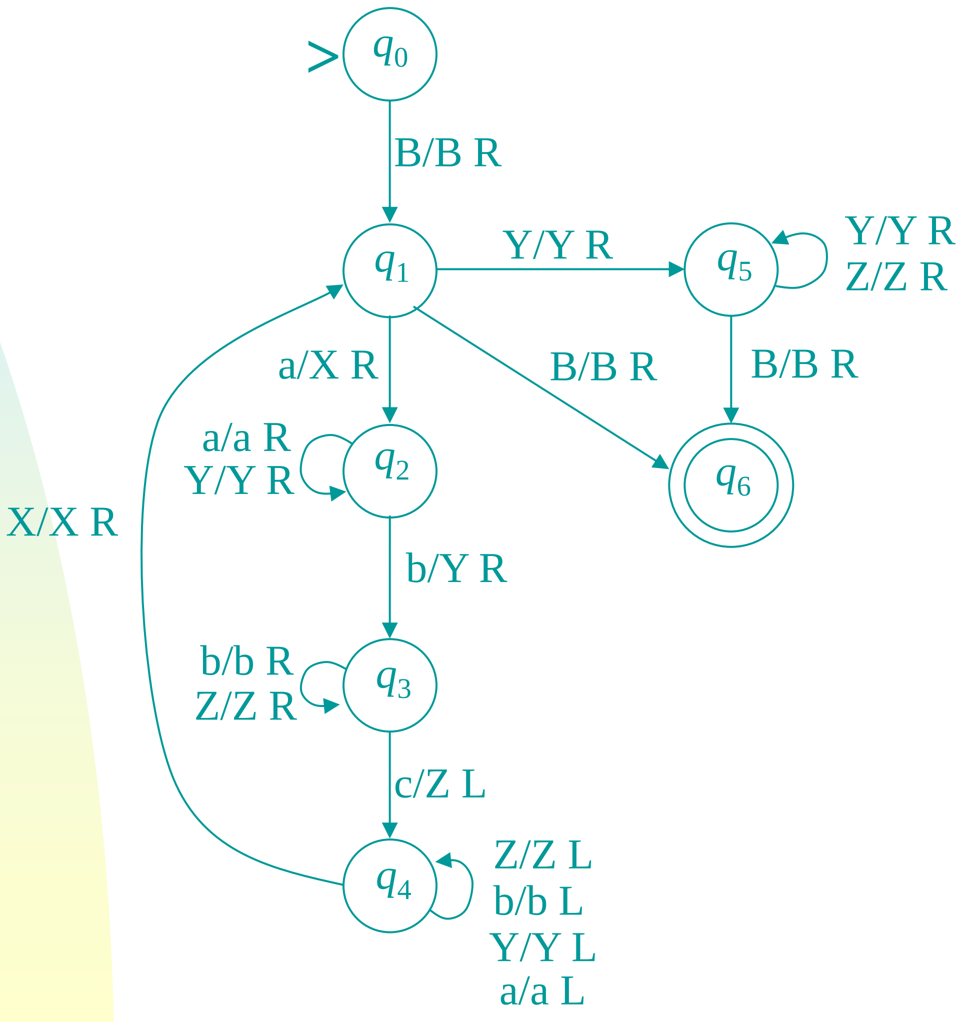
TMs

- Transitions of TMs:
  - $uq_ivB \vdash_{m} xq_jyB$ denotes that xq_jyB is obtained from uq_ivB by a <u>single</u> transition of M
  - $uq_ivB \vdash^{*}_{m} xq_jyB$ denotes that xq_jyB is obtained from uq_ivB by <u>zero</u> or <u>more</u> transitions of M.

- TMs as Language Acceptors
  - TMs can be designed to *accept languages* besides *computing functions*, and accepting a string does <u>not</u> require the entire input string to be read.
  - <u>Defn. 8.2.1</u> Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a TM. A string $u \in \Sigma^*$ is accepted by final state if the computation of M with input u halts in a final state. A computation that terminates abnormally *rejects* the input regardless of the state in which the machine halts. The language of M, $L(M)$, is the set of all string accepted by M. A language accepted by a TM is called a recursively enumerable language.

Transitions of TMs

- Example 8.2.2. A TM that accepts the language { $a^ib^ic^i$ | $i \geq 0$ } is



TMs Acceptance by Halting

- An input string S is accepted by TM M if the computation with S causes M to halt. M rejects S when M terminates abnormally or M never halts with S.

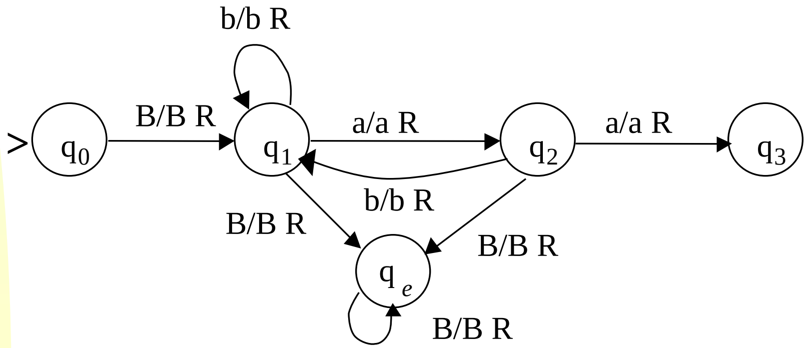
- A TM of which its acceptance is defined by halting (normally) is defined by the quintuple $(Q, \Sigma, \Gamma, \delta, q_0)$.

- Theorem 8.3.2 The following statements are equivalent:

  i) The language L is accepted by a TM that accepts by final state

  ii) The language L is accepted by a TM that accepts by halting

- Example 8.3.1 A TM that accepts $(a \cup b)^*aa(a \cup b)^*$ by halting.



Transitions of TMs

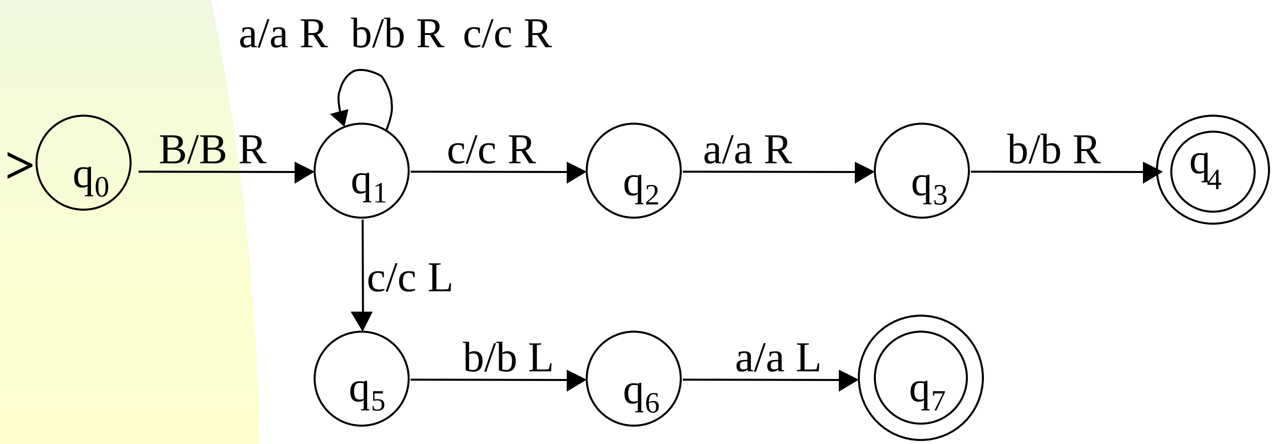
- Design a TM that computes the proper-subtraction function, i.e., $m \dot{-} n = \max(m - n, 0)$ such that $m \dot{-} n$ is $m - n$, if $m > n$, and 0, if $m \leq n$. The TM will start with a tape consisting $0^m 1 0^n$ surrounded by blanks, i.e., $B0^m10^nB$. The machine halts with its result on its tape, surrounded by blanks.

8.7 Nondeterministic TMs (NTMs)

- Provide *more than one applicable transition* for some current state/input symbol pair, i.e., > 1 non-deterministic choice

- Formal Definition: A NTM M = (Q, Σ, Γ, δ, q_0) or M = (Q, Σ, Γ, δ, q_0, F), where Q, Σ, Γ, δ, q_0, and F are as defined in any DTMs and δ: $Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}}$

- Example 8.7.1 A NTM that accepts strings containing a '*c*', which is either *preceded* or *followed* by '*ab*'



8.7 Nondeterministic TMs (NTMs)

- Acceptance in NTMs can be defined by *final state* or by *halting* alone, similar to the DTMs
- Every NTM can be transformed into an equivalent DTM that accepts by *halting*, which is chosen because it reduces the number of computations from 3 to 2
- The language accepted by NTMs are precisely those accepted by DTMs
  - The converted can be done by using multiple (3-) tapes
  - *Multiple* computations for a single input string are sequentially generated and examined
  - A *maximum* number of transitions can be defined for any combination of <state, input symbol>

8.7 Nondeterministic TMs (NTMs)

- Example. Transforming the NTM in Example 8.7.1 into its DTM

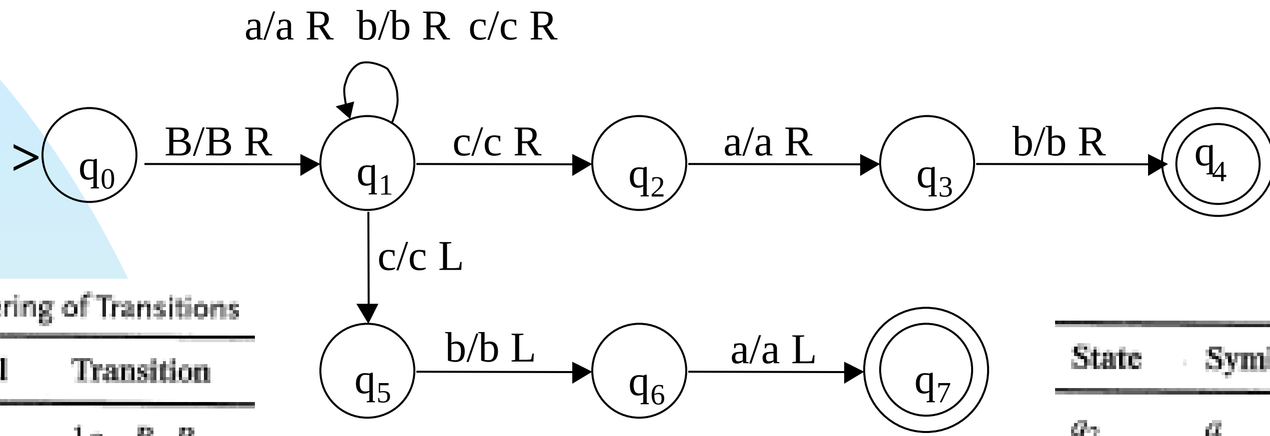


TABLE 8.1 Ordering of Transitions

State	Symbol	Transition
q_0	B	$1 q_1, B, R$
		$2 q_1, B, R$
		$3 q_1, B, R$
q_1	a	$1 q_1, a, R$
		$2 q_1, a, R$
		$3 q_1, a, R$
q_1	b	$1 q_1, b, R$
		$2 q_1, b, R$
		$3 q_1, b, R$
q_1	c	$1 q_1, c, R$
		$2 q_2, c, R$
		$3 q_5, c, L$

$q_0BacabB$ 1
$\vdash Bq_1acabB$ 1
$\vdash Baq_1cabB$ 1
$\vdash Bacq_1abB$ 1
$\vdash Bacaq_1bB$ 1
$\vdash Bacabq_1B$

$q_0BacabB$ 1
$\vdash Bq_1acabB$ 1
$\vdash Baq_1cabB$ 2
$\vdash Bacq_2abB$ 1
$\vdash Bacaq_3bB$ 1
$\vdash Bacabq_4B$

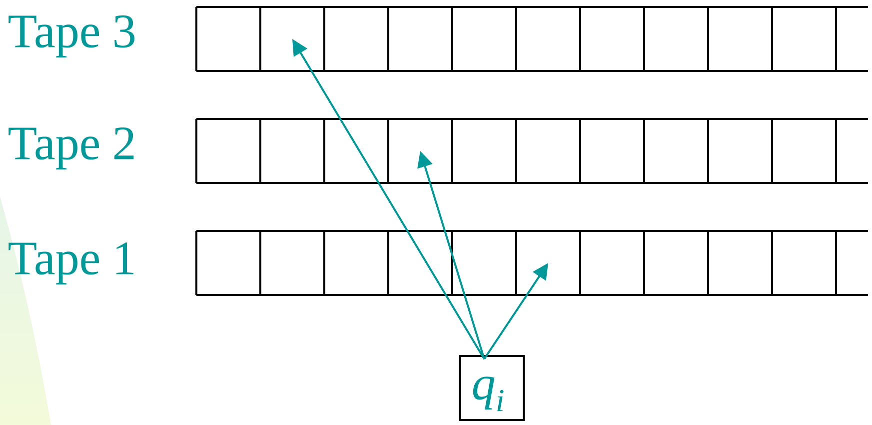
State	Symbol	Transition
q_2	a	$1 q_3, a, R$
		$2 q_3, a, R$
		$3 q_3, a, R$
q_3	b	$1 q_4, b, R$
		$2 q_4, b, R$
		$3 q_4, b, R$
q_5	b	$1 q_6, b, L$
		$2 q_6, b, L$
		$3 q_6, b, L$
q_6	a	$1 q_7, a, L$
		$2 q_7, a, L$
		$3 q_7, a, L$

Transitions of TMs

- Design a TM that takes as input a number *N* and adds 1 to it in *binary*. The tape initially contains a $ followed by *N* in *binary*. The tape head is initially scanning the $ in the initial state q_0. The TM should halt with *N*+1, in *binary*, on the tape, scanning the leftmost symbol of *N*+1 in the final state q_f with $ removed. For instance, $q_0$$10011 yields q_f10100, and $q_0$$11111 yields q_f100000.

8.6 Multitape TMs

- A *K*-tape TM
  - consists of *k tapes* and *k* independent *tape heads*
  - reads *k* tapes simultaneously, but has only <u>one</u> *state*
  - is configured by the *current tape symbol* being pointed to by each tape head and the *current state*, e.g.,



- A transition in a multitape TM may
  - <u>change</u> the current state,
  - <u>(over)write</u> a symbol on each tape, and
  - independently <u>reposition</u> each of the tape heads

8.6 Multitape TMs

- A transition of a *k*-tape TM is defined as

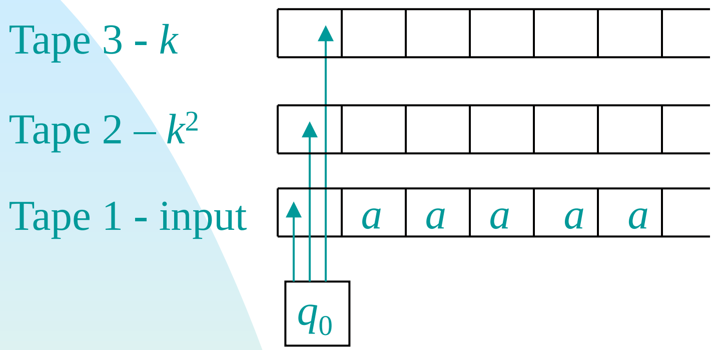
$$\delta(q_i, x_1, x_2, \ldots, x_k) = [q_j; y_1, d_1; y_2, d_2; \ldots; y_k, d_k]$$

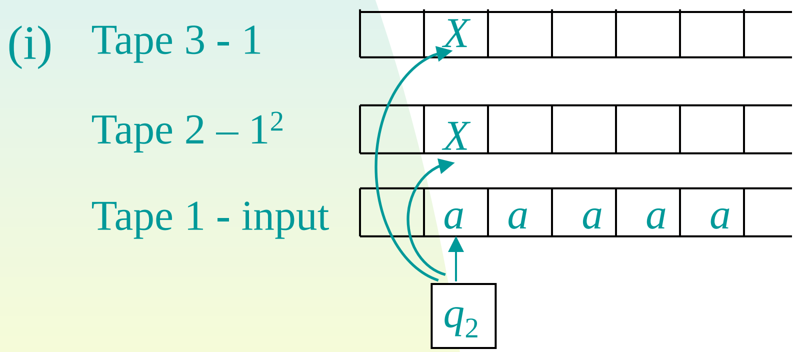
where $q_i, q_j \in Q$, $x_n, y_n \in \Gamma$, and $d_n \in \{ L, R, S \}$, $1 \le n \le k$.

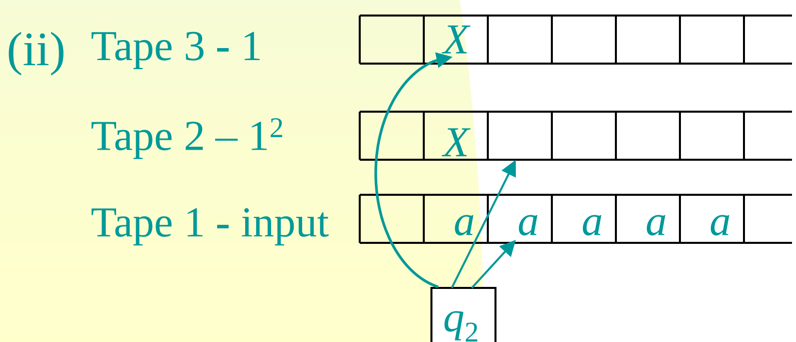
- Initialize configuration:
  - The *input string* is placed on tape 1, whereas all the other tapes are assumed to be *blank* to begin with.
  - The tape heads scan the *leftmost* position of each tape.
  - Any tape head attempts to move to the *left* of the leftmost position terminates the computation *abnormally*.
- A language accepted by a TM is a recursively enumerable language.
- A language that is accepted by a TM that *halts* for all input strings is said to be recursive.

8.6 Multitape TMs

- Example 8.6.2 The set { a^k | k is a *perfect square* } is a *recursively enumerable* language (and is also a *recursive* language).







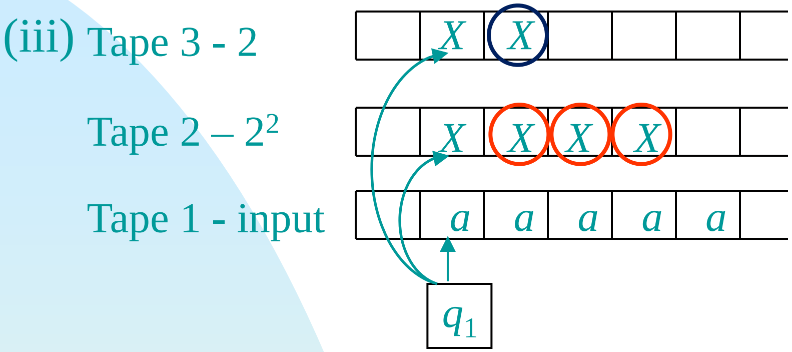
- Tape 1 holds the input string, a string of a's
- Tape 2 holds a string of X's whose length is a **perfect square**
- Tape 3 holds a string of X's whose length is $\sqrt{|S|}$, where S is the string on Tape 2
- Step 1: Since the input is not a null string, initialize tapes 2 and 3 with an X, and all the tape head move to *Position* 1
- Step 2: Move the heads of tapes 1 and 2 to the right, since they have scanned a *nonblank* square

  Accept: if both read a *blank*

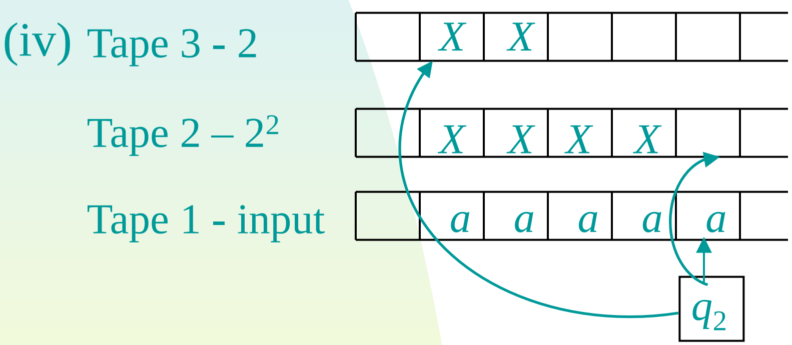
  Reject: if tape head 1 reads a *blank* and tape head 2 reads an X

8.6 Multitape TMs

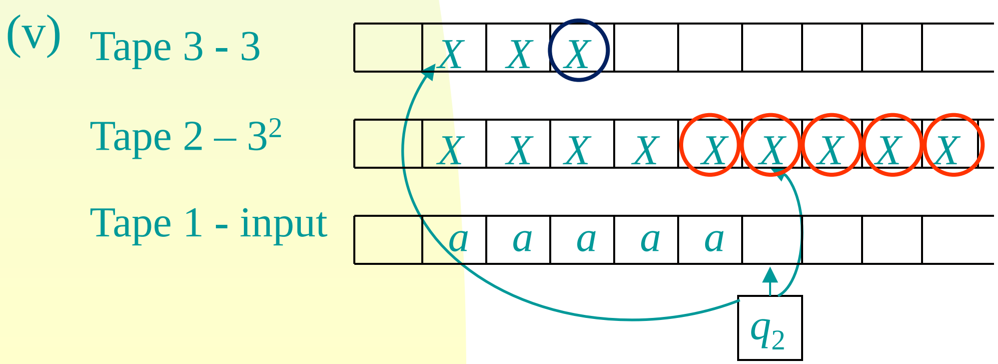
- Example 8.6.2 (Continued).



- Step 3: *Reconfiguration* for comparison with the next perfect square by
  - adding an X on tape 2 to yield k^2+1 X's
  - appending *two* copies of the string on tape 3 to the end of the string on tape 2 to yield $(k+1)^2$ X's
  - adding an X on tape 3 to yield $(k + 1)$ X's on tape 3
  - moving all the tape heads to *Position* 1



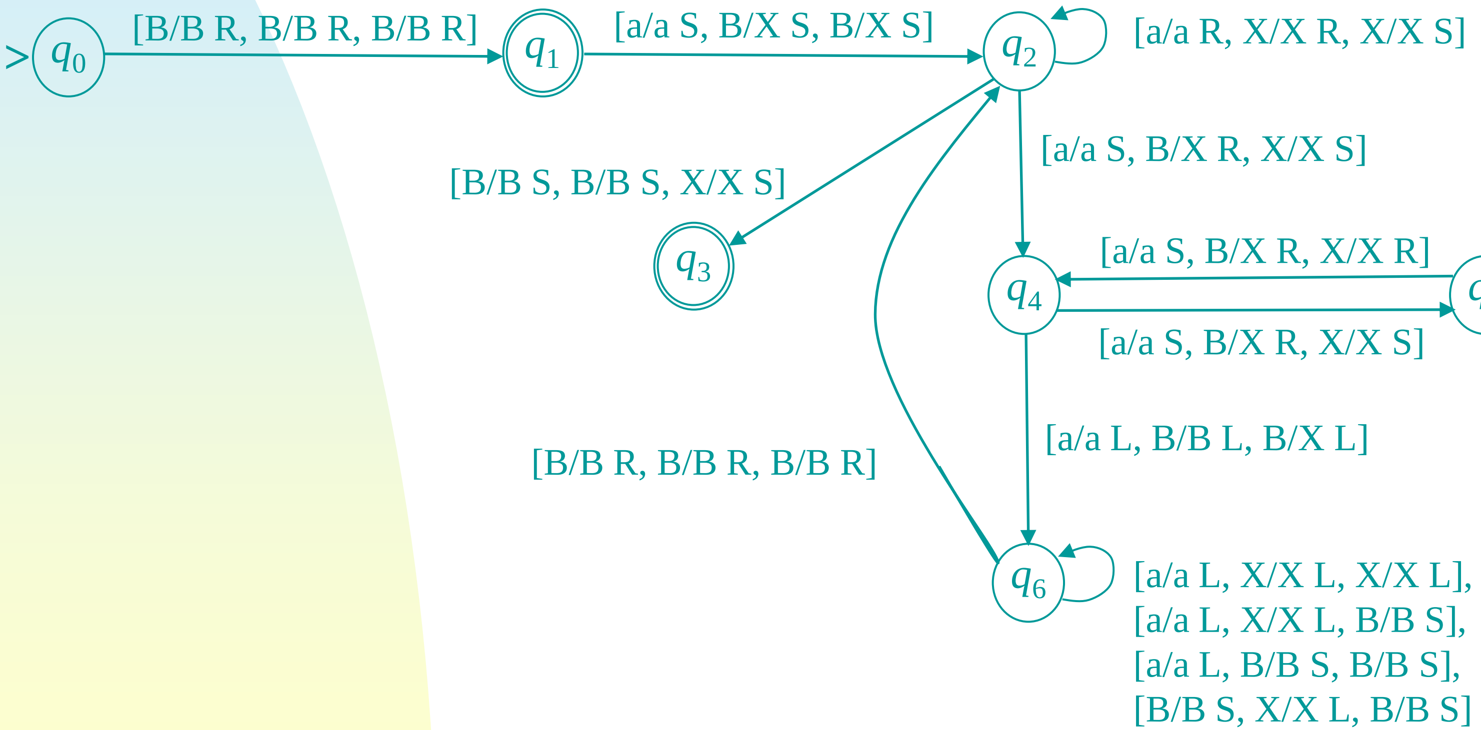
- Step 4: Repeat Steps 2 through 3.



- Another iteration of Step 2 halts and rejects the input.

8.6 Multitape TMs

- Example 8.6.2 (Continued). The transition function of the TM that accepts { a^k | *k* is a *perfect square* }:



8.6 Multitape TMs

- Example 8.6.2 (Continued). The transition function of the TM that accepts { a^k | k is a *perfect square* }:

[Step 1]

$\delta(q_0, B, B, B) = [q_1; B, R; B, R; B, R]$ (*initialize* the tape)
$\delta(q_1, a, B, B) = [q_2; a, S; X, S; X, S]$ (q_1 is a *final* state)

[Step 2]

$\delta(q_2, a, X, X) = [q_2; a, R; X, R; X, S]$ (*compare* strings on tapes 1 and 2)
$\delta(q_2, B, B, X) = [q_3; B, S; B, S; X, S]$ (*accept*; q_3 is a *final* state)
$\delta(q_2, a, B, X) = [q_4; a, S; X, R; X, S]$ (*add* an X to tape 2 and re-compute)

[Step 3]

$\delta(q_4, a, B, X) = [q_5; a, S; X, R; X, S]$ (*rewrite* tapes 2 and 3)
$\delta(q_5, a, B, X) = [q_4; a, S; X, R; X, R]$ (add two X's to tape 2 for each X on tape 3 – to generate $(k+1)^2$)
$\delta(q_4, a, B, B) = [q_6; a, L; B, L; X, L]$ (add an X's to tape 2 to yield k+1)

[Step 4]

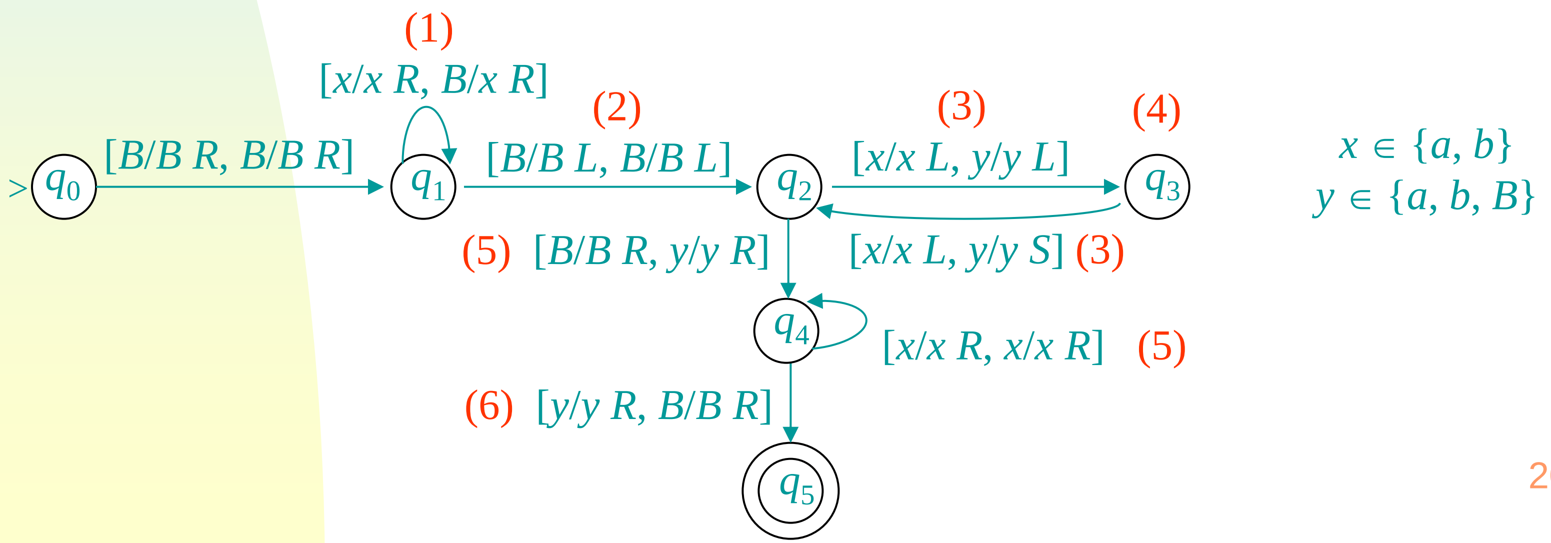
$\delta(q_6, a, X, X) = [q_6; a, L; X, L; X, L]$ (*reposition* tape heads)
$\delta(q_6, a, X, B) = [q_6; a, L; X, L; B, S]$ (tape 3 at 1st cell, but not tapes 1 & 2)
$\delta(q_6, a, B, B) = [q_6; a, L; B, S; B, S]$ (tape 2 & 3 at 1st cell, but not tape 1)
$\delta(q_6, B, X, B) = [q_6; B, S; X, L; B, S]$ (tape 1 & 3 at 1st cell, but not tape 2)
$\delta(q_6, B, B, B) = [q_2; B, R; B, R; B, R]$ (*repeat* comparison cycle)

8.6 Multitape TMs

- A multitape TM can be represented by a state transition diagram.
- Example 8.6.3 A 2-tape TM that accepts $\{ uu \mid u \in \{ a, b \}^* \}$.

Computation:

1) Make a *copy* of the input *S* (on tape 1) to tape 2; tape heads: right of *S*.
2) Move both tape heads one step to the left.
3) Move the head of tape 1 *two* squares for each square move of tape 2.
4) *Reject* the input *S* if the TM halts in q_3. (i.e. $|S|$ is *odd*.)
5) *Compare* the 1st half with the 2nd half of *S* in q_4
6) *Accept S* in q_5



8.6 Multitape TMs

- Theorem 8.6.1 A language *L* is accepted by a multitape TM iff it is accepted by a standard TM.

  Proof. By simulating a multitape TM using a single tape with multitracksTM.