Chapter 7

PDA and CFLs

7.1 PDA

- Is an enhanced FSA with an internal memory system, i.e., a (*pushdown*) *stack*.

- Overcomes the memory limitations and *increases* the *processing power* of FSAs.

- Defn. 7.1.1 A pushdown automaton (PDA) is a sextuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where
  - Q is a finite set of states
  - Σ is a finite set of input symbols, called input alphabet
  - Γ is a finite set of stack symbols, called stack alphabet
  - $q_0 \in Q$, is the start state
  - $F \subseteq Q$, is the set of final states
  - δ: $Q \times (\Sigma \cup \{ \lambda \}) \times (\Gamma \cup \{ \lambda \}) \rightarrow Q \times (\Gamma \cup \{ \lambda \})$, a (*partial*) transition function

7.1 PDA

- A convention:
  - Stack symbols are *capital letters*
  - Greek letters represent *strings* of *stack symbols*
  - An empty stack is denoted λ
  - $A\alpha$ represents a stack with *A* as the *top element*

7.1 PDA

- δ is of the form

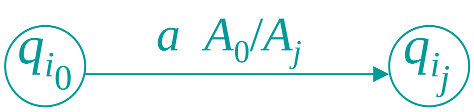
$$\delta(q_{i_0}, a, A_0) = \{ [q_{i_1}, A_1], [q_{i_2}, A_2], \ldots, [q_{i_n}, A_n] \}$$

where the transition

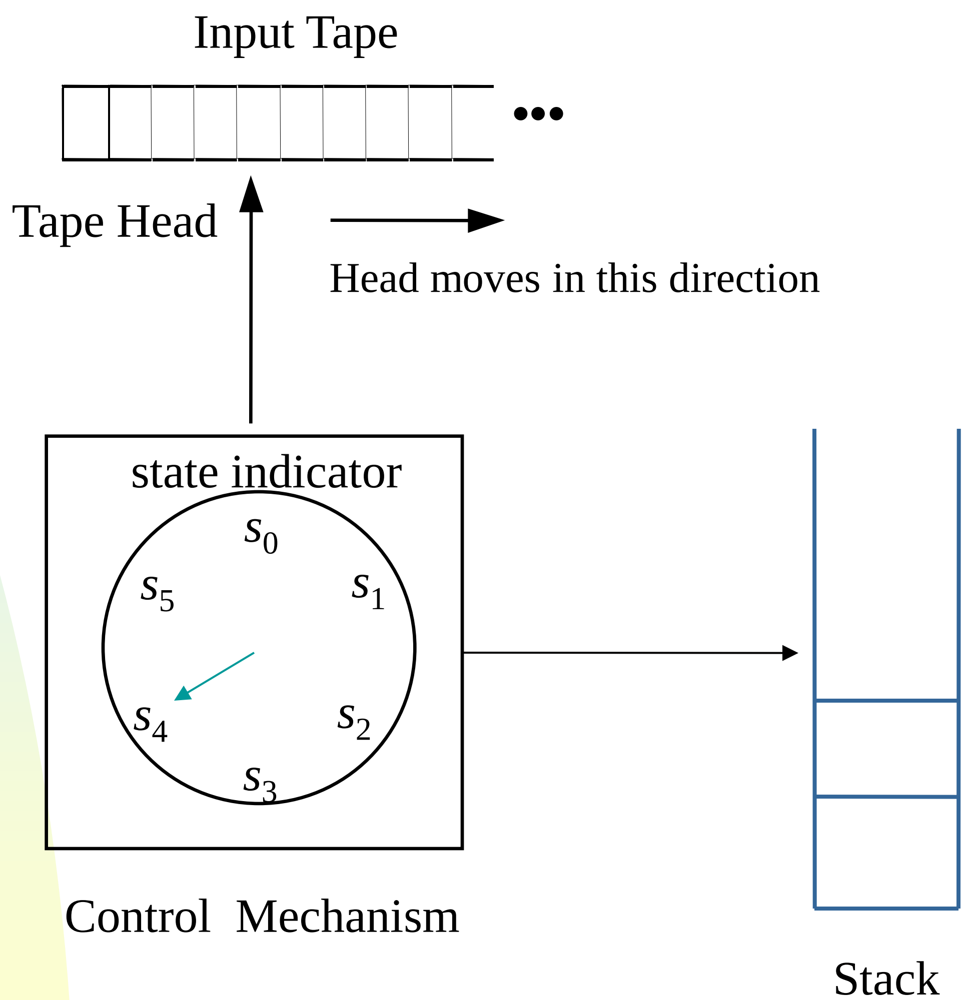
$$[q_{i_j}, A_j] \in \delta(q_{i_0}, a, A_0),\ 1 \le j \le n$$

denotes that

  - q_{i_0} is the *current state*
  - a is the *current input symbol*
  - A_0 is the current *top* of the *stack symbol*
  - q_{i_j} $(1 \le j \le n)$ is the *new state*, and
  - A_j is the *new top* of the *stack symbol* and in a state (transition) diagram, it is denoted



7.1 Pushdown Automaton

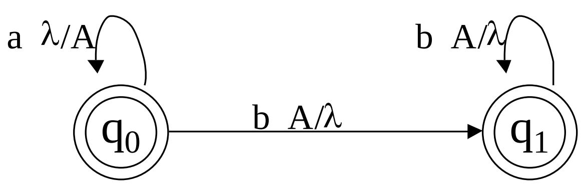


7.1 PDA

- Special cases (note that q_i and q_j can be the same):
  - $[q_j, A] \in \delta(q_i, a, \lambda)$ /* Consume the input, push a stack symbol */
  - $[q_j, \lambda] \in \delta(q_i, \lambda, A)$ /* Consume no input, pop the TOS symbol */
  - $[q_j, A] \in \delta(q_i, \lambda, \lambda)$ /* Consume no input, push a stack symbol */
  - $[q_j, \lambda] \in \delta(q_i, a, \lambda)$ /* Consume input, no push/pop, an FSA transition */

- The PDA notation $[q_i, w, \alpha] \overset{*}{\underset{m}{\vdash}} [q_j, v, \beta]$ indicates that $[q_j, v, \beta]$ can be obtained from $[q_i, w, \alpha]$ as a result of a *sequence* of transitions (i.e., 0 or more).

- <u>Example</u>. The PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$, where δ is



Accepts $a^n b^n$ ($n \geq 0$) with an *empty stack* and in an *accepting state*

7.1 PDA

- Defn. 7.1.2 Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA. A string $w \in \Sigma^*$ is accepted by M if $\exists\ [q_0, w, \lambda] \vdash^* [q_i, \lambda, \lambda]$, where $q_i \in F$. $L(M)$, the language of M, is the set of strings accepted by M.

- Example (7.1.1) Give a PDA that accepts the language $\{ wcw^R \mid w \in \{ a, b \}^* \}$.

- Defn. A PDA is *deterministic* if there is at most one transition that is applicable for each configuration of <*state*, *input symbol*, *stack top symbol*>.
  - Example 7.1.2 Construct $L = \{ a^i \mid i \geq 0 \} \cup \{ a^i b^i \mid i \geq 0 \}$
  - Example 7.1.3 Construct all even-length palindromes over $\{ a, b \}$
  - Nondeterministic PDAs allow the machines to “guess”
  - For some NPDAs, their counterparts (i.e., DPDAs) do not exist
  - Languages L accepted by DPDA include RL, and $L \subset$ CFL

7.2 Variations on PDAs

- Defn. A PDA is *atomic* if each transition in the PDA is of one of the following forms:

  $[q_j, \lambda] \in \delta(q_i, a, \lambda)$: *process* an input symbol

  $[q_j, \lambda] \in \delta(q_i, \lambda, A)$: *pop* the stack

  $[q_j, A] \in \delta(q_i, \lambda, \lambda)$: *push* a stack symbol

- Theorem 7.2.1. Let M be a PDA. Then $\exists$ an atomic PDA M' such that $L(M') = L(M)$.

  - Replace each *non-atomic* transition by a sequence of *atomic* transitions

- Defn. A transition $[q_j, \alpha] \in \delta(q_i, a, A)$, where $\alpha \in \Gamma^+$ is called *extended transition*. A PDA containing extended transition is called an extended PDA.

  - Example 7.2.1. Construct $L = \{ a^i b^{2i} \mid i \geq 1 \}$, with PDA, atomic PDA, and extended PDA

7.2 Variations on PDAs

Example 7.2.1

Let $L = \{a^i b^{2i} \mid i \geq 1\}$. A PDA, an atomic PDA, and an extended PDA are constructed to accept L. The input alphabet $\{a, b\}$, stack alphabet $\{A\}$, and accepting state q_1 are the same for each automaton.

PDA

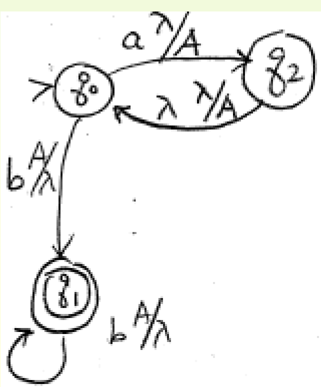
$Q = \{q_0, q_1, q_2\}$

$\delta(q_0, a, \lambda) = \{[q_2, A]\}$

$\delta(q_2, \lambda, \lambda) = \{[q_0, A]\}$

$\delta(q_0, b, A) = \{[q_1, \lambda]\}$

$\delta(q_1, b, A) = \{[q_1, \lambda]\}$



Atomic PDA

$Q = \{q_0, q_1, q_2, q_3, q_4\}$

$\delta(q_0, a, \lambda) = \{[q_3, \lambda]\}$

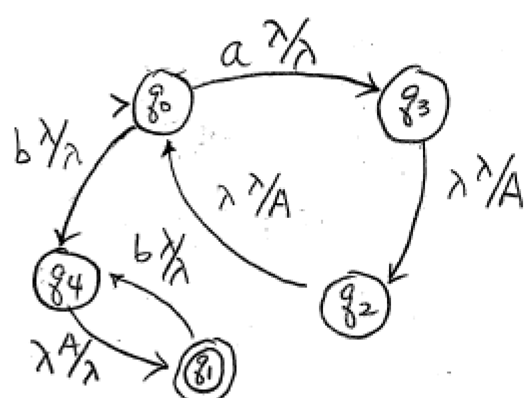
$\delta(q_3, \lambda, \lambda) = \{[q_2, A]\}$

$\delta(q_2, \lambda, \lambda) = \{[q_0, A]\}$

$\delta(q_0, b, \lambda) = \{[q_4, \lambda]\}$

$\delta(q_4, \lambda, A) = \{[q_1, \lambda]\}$

$\delta(q_1, b, \lambda) = \{[q_4, \lambda]\}$



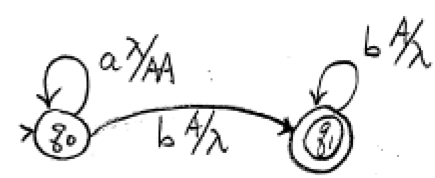
Extended PDA

$Q = \{q_0, q_1\}$

$\delta(q_0, a, \lambda) = \{[q_0, AA]\}$

$\delta(q_0, b, A) = \{[q_1, \lambda]\}$

$\delta(q_1, b, A) = \{[q_1, \lambda]\}$



PDA

- Defn. A string w is accepted by final state if $\exists$ a computation $[q_0, w, \lambda] \vdash^* [q_i, \lambda, \alpha]$, where $q_i \in F$ and $\alpha \in \Gamma^*$, i.e., the content of the stack is *irrelevant*.

- Lemma 7.2.3. Let L be a language accepted by a PDA M with acceptance defined by *final state*. Then $\exists$ a PDA M' that accepts L by *final state* and *empty stack*.

- Proof. Let $M' = (Q \cup \{q_f\}, \Sigma, \Gamma, \delta', q_0, \{q_f\})$, where

  $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$,

  $\delta' \supseteq \delta$,

  $\forall q_i \in F, \delta'(q_i, \lambda, \lambda) = \{ [q_f, \lambda] \}$, and

  $\forall A \in \Gamma, \delta'(q_f, \lambda, A) = \{ [q_f, \lambda] \}$

  hence, given M,

  $$[q_0, w, \lambda] \underset{M}{\overset{*}{\vdash}} [q_i, \lambda, \alpha] \underset{M'}{\vdash} [q_f, \lambda, \alpha] \underset{M'}{\overset{*}{\vdash}} [q_f, \lambda, \lambda].$$

PDA

- Defn. A string w is said to be accepted by empty stack if $\exists$ a computation $[q_0, w, \lambda] \vdash^{+} [q_i, \lambda, \lambda]$, where q_i may *not* be a final state.

- Lemma 7.2.4. Let L be a language accepted by a PDA M with acceptance defined by *empty stack*. Then $\exists$ a PDA M' that accepts L by *final state* and *empty stack*.

Proof. (P. 230) Let $M' = (Q \cup \{q_0'\}, \Sigma, \Gamma, \delta', q_0', Q)$, where

$$M = (Q, \Sigma, \Gamma, \delta, q_0), \text{ and}$$

each state in M' is a *final state*, except q_0', the new *start state*, where

$$\delta'(q_0', a, A) = \delta(q_0, a, A), \text{ and}$$

$$\forall q_i \in Q, a \in \Sigma \cup \{ \lambda \}, A \in \Gamma \cup \{ \lambda \},$$

$$\delta'(q_i, a, A) = \delta(q_i, a, A)$$

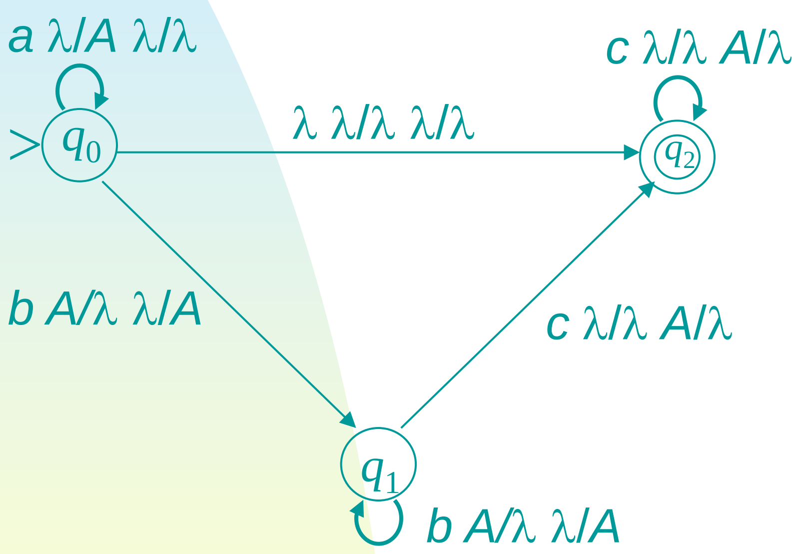
Two-Stack PDAs

- Two-stack PDAs, an extension of PDAs
- A two-stack PDA (2PDA) is a sextuple (Q, Σ, Γ, δ, q_0, F), where
  - Q, Σ, Γ, q_0, and F are the same as in a one-stack PDA
  - δ: $Q \times (\Sigma \cup \{ \lambda \}) \times (\Gamma \cup \{ \lambda \}) \times (\Gamma \cup \{ \lambda \}) \rightarrow$
    $Q \times (\Gamma \cup \{ \lambda \}) \times (\Gamma \cup \{ \lambda \})$
- 2PDAs accept non-CFLs, in addition to all CFLs
- Accepting criteria:
  - Consume an *input string*
  - Enter a *final state*
  - *Empty* both *stacks*

Two-Stack PDAs

- Example. Given $L = \{ a^i b^i c^i \mid i \geq 0 \}$, L is not a CFL.

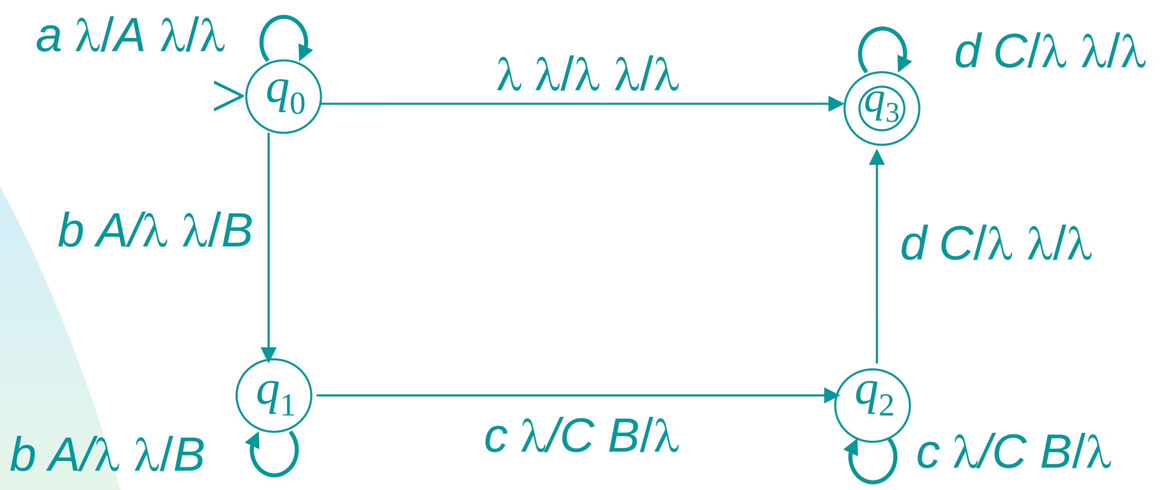
  A 2PDA M that accepts L is



$$
\begin{aligned}
[q_0, aabbcc, \lambda, \lambda] &\vdash [q_0, abbcc, A, \lambda] \\
&\vdash [q_0, bbcc, AA, \lambda] \\
&\vdash [q_1, bcc, A, A] \\
&\vdash [q_1, cc, \lambda, AA] \\
&\vdash [q_2, c, \lambda, A] \\
&\vdash [q_2, \lambda, \lambda, \lambda]
\end{aligned}
$$

Two-Stack PDAs

- Example. A 2PDA M accepts $L = \{ a^i b^i c^i d^i \mid i \geq 0 \}$



$[q_0, aabbccdd, \lambda, \lambda] \vdash [q_0, abbccdd, A, \lambda]$
$\vdash [q_0, bbccdd, AA, \lambda]$
$\vdash [q_1, bccdd, A, B]$
$\vdash [q_1, ccdd, \lambda, BB]$
$\vdash [q_2, cdd, C, B]$
$\vdash [q_2, dd, CC, \lambda]$
$\vdash [q_3, d, C, \lambda]$
$\vdash [q_3, \lambda, \lambda, \lambda]$

7.3 PDA and CFLs

- <u>Defn. 5.6.1.</u> A CFG $G = (V, \Sigma, P, S)$ is in Greibach normal form (*GNF*) if each rule has one of the following forms:

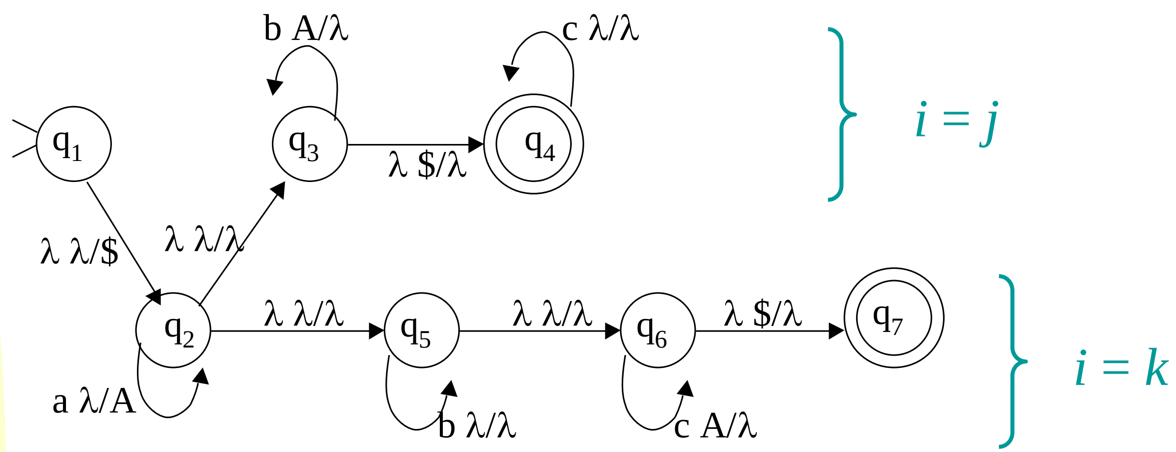
  i) $A \rightarrow aA_1A_2 \ldots A_n$

  ii) $A \rightarrow a$

  iii) $S \rightarrow \lambda$

  where $a \in \Sigma$ and $A_i \in V - \{ S \}$, $i = 1, 2, \ldots, n$

- <u>Example</u>: Given the language $L = \{ a^ib^jc^k \mid i, j, k \geq 0$ and $(i = j$ or $i = k) \}$, the following PDA accepts L:



7.3 PDA and CFLs

- The CFG G that generates the set of string in the language

  $L = \{ a^i b^j c^k \mid i, j, k \geq 0$ and $(i = j$ or $i = k) \}$, i.e., L(G), is

$$
\begin{array}{rl}
 & S \rightarrow aAc \mid aDbC \mid \lambda \mid B \mid C \\
i = k \{ & A \rightarrow aAc \mid bB \mid \lambda \\
 & B \rightarrow bB \mid \lambda \\
i = j \{ & D \rightarrow aDb \mid \lambda \\
 & C \rightarrow cC \mid \lambda
\end{array}
$$

7.3 PDA and CFLs

- <u>Theorem 7.3.1</u> Let L be a CFL. Then $\exists$ a PDA that accepts L.

- <u>Proof</u>. Let $G = (V, \Sigma, P, S)$ be a grammar in GNF that generates L. An extended PDA M with start state q_0 is defined by

  $Q_m = \{ q_0, q_1 \}$, $\Sigma_m = \Sigma$, $\Gamma_m = V - \{ S \}$, and $F_m = \{ q_1 \}$

  with transitions

  (a) $\delta(q_0, a, \lambda) = \{ [q_1, w] \mid S \to aw \in P \}$

  (b) $\delta(q_1, a, A) = \{ [q_1, w] \mid A \to aw \in P \text{ and } A \in V - \{ S \}\}$

  (c) $\delta(q_0, \lambda, \lambda) = \{ [q_1, \lambda] \mid S \to \lambda \in P \}$

7.3 PDA and CFLs

Proof. We must show that

i) $L \subseteq L(M)$

For each derivation $S \overset{*}{\Rightarrow} uw$ with $u \in \Sigma^+$ and $w \in V^*$, we show that $\exists$ a computation $[q_0, u, \lambda] \overset{*}{\vdash} [q_1, \lambda, w]$ in M by induction on the length of the derivation, i.e., n.

Basis: $n = 1$, i.e., $S \Rightarrow aw$, where $a \in \Sigma$ and $w \in V^*$

The transition (a), i.e., $\delta(q_0, a, \lambda) = \{ [q_1 , w] \mid S \to aw \in P \}$, yields the desired computation.

Induction Hypothesis:

Assume for every derivation $S \overset{n}{\Rightarrow} uw$, $\exists$ a computation $[q_0, u, \lambda] \overset{*}{\vdash} [q_1, \lambda, w]$ in M.

PDA and CFLs

Induction:

Now consider $S \overset{n+1}{\Rightarrow} uw$. Let $u = va \in \Sigma^+$ & $w \in V^*$, $S \overset{n+1}{\Rightarrow} uw$ can be written as $S \overset{n}{\Rightarrow} vAw_2 \Rightarrow uw$, where $w = w_1w_2$ & $A \rightarrow aw_1 \in P$.

By I.H. & $[q_1, w_1] \in \delta(q_1, a, A)$ of Transition (b), i.e.,

$$\delta(q_1, a, A) = \{ [q_1, w] \mid A \rightarrow aw \in P \;\&\; A \in V - \{ S \} \}$$

$$[q_0, va, \lambda] \overset{*}{\vdash} [q_1, a, Aw_2]$$
$$\vdash [q_1, \lambda, w_1w_2]$$

If $\lambda \in L$, then $S \rightarrow \lambda \in P$ yields the Transition (c), i.e.,

$$[q_0, \lambda, \lambda] \vdash [q_1, \lambda]$$

- ii) $L(M) \subseteq L$

  Show that for every computation $[q_0, u, \lambda] \overset{*}{\vdash} [q_1, \lambda, w]$, $\exists$ a derivation $S \overset{*}{\Rightarrow} uw$ in G by induction.

PDA and CFLs

- Every language accepted by a PDA is *context-free*
- The *Transformation Algorithm*:

  Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA. We construct the grammar G .э. $L(G) = L(M)$.

  (i) Construct an extended PDA M' w/ δ' as its transition function from M .э.

  (a) Given $[q_j, \lambda] \in \delta(q_i, u, \lambda)$, construct $[q_j, A] \in \delta'(q_i, u, A)$, $\forall A \in \Gamma$

  (b) Given $[q_j, B] \in \delta(q_i, u, \lambda)$, construct $[q_j, BA] \in \delta'(q_i, u, A)$, $\forall A \in \Gamma$

  (ii) Given the PDA M' as constructed in step (i), construct $G = (V, \Sigma, P, S)$, where $V = \{ S \} \cup \{ <q_i, A, q_j> \mid q_i, q_j \in Q, A \in \Gamma \cup \{ \lambda \} \}$.

  - $<q_i, A, q_j>$ denotes a computation that begins in q_i, ends in q_j & removes A ($\in \Gamma$) from the stack.

The Transformation Algorithm

- P is constructed as follows:

1. $S \to \langle q_0, \lambda, q_j \rangle, \forall q_j \in F$

2. For each transition $[q_j, B] \in \delta(q_i, X, A)$, where $A \in \Gamma \cup \{\lambda\}$,
 create $\{\langle q_i, A, q_k \rangle \to X\langle q_j, B, q_k \rangle \mid q_k \in Q\}$

3. For each transition $[q_j, BA] \in \delta(q_i, X, A)$, where $A \in \Gamma$,
 create $\{\langle q_i, A, q_k \rangle \to X\langle q_j, B, q_n \rangle\langle q_n, A, q_k \rangle \mid q_k, \mathrm{q_n} \in Q\}$

4. For each $q_k \in Q$, create $\langle q_k, \lambda, q_k \rangle \to \lambda$.

Rule 1: A computation begins w/ the *start state*, ends in a *final state*, & terminate w/ an *empty stack*, i.e., a <u>successful computation</u> in *M*'

Rules 2 & 3: Trace the transitions of *M*'

Rule 4: Terminate derivations

PDA & CFLs

- <u>Example 7.3.1.</u> Given the PDA M such that L(M) = { $a^n cb^n \mid n \geq 0$ }. The corresponding CFG G is given in Table 7.3.1 (on P.240).

- M is

  $Q = \{q_0, q_1\}$ $\quad\quad$ $\delta(q_0, a, \lambda) = \{[q_0, A]\}$

  $\Sigma = \{a, b, c\}$ $\quad\quad$ $\delta(q_0, c, \lambda) = \{[q_1, \lambda]\}$

  $\Gamma = \{A\}$ $\quad\quad$ $\delta(q_1, b, A) = \{[q_1, \lambda]\}$

  $F = \{q_1\}$

- M’ is M with the additional transitions:

  $\delta(q_0, a, A) = \{ [q_0, AA] \}$ and

  $\delta(q_0, c, A) = \{ [q_1, A] \}$

PDA and CFLs

- Example 7.3.1 P in G includes:
  - using Rule 1: $S \to < q_0, \lambda, q_1 >$
  - given $\delta(q_0, a, \lambda) = \{ [q_0, A] \}$ & Rule 2: $<q_0, \lambda, q_0> \to a <q_0, A, q_0>$
    $<q_0, \lambda, q_1> \to a <q_0, A, q_1>$
  - given $\delta(q_0, a, A) = \{[q_0, AA]\}$ & Rule 3:
    $<q_0, A, q_0> \to a <q_0, A, q_0> <q_0, A, q_0 >$
    $<q_0, A, q_0> \to a <q_0, A, q_1> <q_1, A, q_0 >$
    $<q_0, A, q_1> \to a <q_0, A, q_0> <q_0, A, q_1 >$
    $<q_0, A, q_1> \to a <q_0, A, q_1> <q_1, A, q_1 >$
  - given $\delta(q_0, c, \lambda) = \{ [q_1, \lambda] \}$ & Rule 2:
    $<q_0, \lambda, q_0> \to c <q_1, \lambda, q_0>$
    $<q_0, \lambda, q_1> \to c <q_1, \lambda, q_1>$
  - given $\delta(q_0, c, A) = \{ [q_1, A] \}$ & Rule 2:
    $<q_0, A, q_0> \to c <q_1, A, q_0>$
    $<q_0, A, q_1> \to c <q_1, A, q_1>$
  - given $\delta(q_1, b, A) = \{ [q_1, \lambda] \}$
    $<q_1, A, q_0> \to b <q_1, \lambda, q_0>$
    $<q_1, A, q_1> \to b <q_1, \lambda, q_1>$
  - using Rule 4: $<q_0, \lambda, q_0> \to \lambda$
    $<q_1, \lambda, q_1> \to \lambda$