Discussion #28

Partial Orders
Topics

• Weak and strict partially ordered sets (posets)
• Total orderings
• Hasse diagrams
• Bounded and well-founded posets
Partial Orders

- Total orderings: single sequence of elements
- Partial orderings: some elements may come before/after others, but some need not be ordered
- Examples of partial orderings:
  - “must be completed before”
  - “set inclusion, $\subseteq$”

```
  foundation
   /       /
framing  wiring
  |       |
plumbing finishing
```

```
{a, b, c}
  /   /
{a, b} {a, c} {b, c}
 /   /
{a}  {b}  {c}
   /   |
  $\emptyset$
```
Partial Order Definitions
(Poset Definitions)

• A relation $R: S \leftrightarrow S$ is called a (weak) partial order if it is reflexive, antisymmetric, and transitive.

• A relation $R: S \leftrightarrow S$ is called a strict partial order if it is irreflexive, antisymmetric, and transitive.

  e.g. $\leq$ on the integers

  e.g. $<$ on the integers
Total Order

• A total ordering is a partial ordering in which every element is related to every other element. (This forces a linear order or chain.)

• Examples:
  
  R: ≤ on \{1, 2, 3, 4, 5\} is total.

  Pick any two; they’re related one way or the other with respect to ≤.

  R: ⊇ on \{{a, b}, \{a\}, \{b\}, \emptyset\} is not total.

  We can find a pair not related one way or the other with respect to ⊇.

  \{a\} & \{b\}: neither \{a\} ⊇ \{b\} nor \{b\} ⊇ \{a\}
Hasse Diagrams

We produce Hasse Diagrams from directed graphs of relations by doing a transitive reduction plus a reflexive reduction (if weak) and (usually) dropping arrowheads (using, instead, “above” to give direction)

1) Transitive reduction — discard all arcs except those that “directly cover” an element.

2) Reflexive reduction — discard all self loops.

For $\supseteq$

\[
\begin{array}{c}
\{a\} \\
\downarrow \\
\{a, b\} \\
\downarrow \\
\emptyset \\
\end{array}
\quad \equiv \quad
\begin{array}{c}
\{a\} \\
\downarrow \\
\{a, b\} \\
\downarrow \\
\emptyset \\
\end{array}
\]

we write:

\[
\begin{array}{c}
\{a\} \\
\downarrow \\
\{b\} \\
\downarrow \\
\emptyset \\
\end{array}
\quad \equiv \quad
\begin{array}{c}
\{a\} \\
\downarrow \\
\{b\} \\
\downarrow \\
\emptyset \\
\end{array}
\]

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Descending Sequence

• Descending sequence: A sequence \(<x_1, x_2, \ldots, x_n>\) where for \(i < j\), \(x_i\) “is strictly above” \(x_j\) on a path in a Hasse diagram; \(x_i\) need not, however, be “immediately above” \(x_j\).

• Examples:

\[
\begin{align*}
\{a,b,c\}, \{c\}, \emptyset & \quad \text{descending} \\
\{a,b,c\}, \{b\}, \{c\}, \emptyset & \quad \text{not descending} \\
\{a,b,c\}, \{b,c\}, \{c\}, \emptyset & \quad \text{descending}
\end{align*}
\]

\[
\begin{align*}
5, 4, 2 & \quad \text{descending} \\
3, 2, 2, 2, 1 & \quad \text{not descending}
\end{align*}
\]
Well Founded Poset

• A poset is *well founded* if it has no infinite descending sequence.

• Examples:
  > on the integers?
    \( <3, 2, 1, 0, -1, \ldots> \) not well founded
  \( 
  \geq \) on finite sets?
    \( <\{a, b, c\}, \{c\}, \emptyset> \) well founded
    All finite strict posets are well founded.
  \( \geq \) on finite sets?
    \( <\{a\}, \{a\}, \{a\}, \ldots> \) not a descending sequence
    All finite (weak) posets are well founded.
  > natural numbers?
    \( <\ldots, 3, 2, 1, 0> \) infinite, but well founded
Application of Well Founded Posets

• Has anyone ever gotten into an infinite loop in a program?

• We use well founded sets to prove that loops terminate.
  
  e.g. The following clearly terminates.
  
  for i=1 to n do …
  
  n–i for i=1, …, n is a descending sequence on a well founded set (the natural numbers): <n–1, n–2, …, n–n = 0>.
## More Interesting Termination Example

//Reachable in a grammar
S' := ∅
S := {rule #'s of start symbol}
while |S| > |S'|
  S' := S
  S := S' ∪ {rule #'s of rhs non-t’s}

| iteration | S       | S'    | #rules | |S|  | |S'| |
|-----------|---------|-------|--------|------|------|------|
| 0.        | {1,2}   | ∅     | 7      | 2    | 0    | 7    |
| 1.        | {1,2,5,7} | {1,2} | 7      | 4    | 2    | 5    |
| 2.        | {1,2,5,7,6} | {1,2,5,7} | 7 | 5    | 4    | 3    |
| 3.        | {1,2,5,7,6} | {1,2,5,7,6} | 7 | 5    | 5    | 2    |

well founded: no infinite descending sequence
no matter what grammar is input.
Upper and Lower Bounds

• If a poset is built from relation R on set A, then any \( x \in A \) satisfying \( xRy \) is an upper bound of \( y \), and any \( x \in A \) satisfying \( yRx \) is a lower bound of \( y \).

• Examples: If \( A = \{a, b, c\} \) and R is \( \supseteq \), then \( \{a, c\} \) is an upper bound of \( \{a\}, \{c\}, \) and \( \emptyset \).
  - is also an upper bound of \( \{a, c\} \) (weak poset).
  - is a lower bound of \( \{a, b, c\} \).
  - is also a lower bound of \( \{a, c\} \) (weak poset).
Maximal and Minimal Elements

• If a poset is built from relation $R$ on set $A$, then $y \in A$ is a \textit{maximal} element if there is no $x$ such that $xRy$, and $x \in A$ is a \textit{minimal} element if there is no $y$ such that $xRy$. (Note: We either need the poset to be strict or $x \neq y$.)

• In a Hasse diagram, every element with no element “above” it is a maximal element, whereas every element with no element “below” it is a minimal element.
Least Upper and Greatest Lower Bounds

- A least upper bound of two elements $x$ and $y$ is a minimal element in the intersection of the upper bounds of $x$ and $y$.
- A greatest lower bound is a maximal element in the intersection of the lower bounds of $x$ and $y$.
- Examples:
  - For $\supseteq$, \{a, c\} is a least upper bound of \{a\} and \{c\}, $\emptyset$ is a greatest lower bound of \{a\} and \{b, c\}, and \{a\} is a least upper bound of \{a\} and $\emptyset$.
  - For the following strict poset, lub(x,y) = \{a,b\}, lub(y,y) = \{a,b,c\}, lub(a,y) = $\emptyset$, glb(a,b) = \{x,y\}, glb(a,c) = \{y\}