Discussion #26

Binary Relations: Operations & Properties
Topics

• Inverse (converse)
• Set operations
• Extensions and restrictions
• Compositions
• Properties
  – reflexive, irreflexive
  – symmetric, antisymmetric, asymmetric
  – transitive
Inverse
(sometimes called converse)

• If $R: A \leftrightarrow B$, then the inverse of $R$ is $R^\sim: B \leftrightarrow A$.
• $R^{-1}$ is also a common notation
• $R^\sim$ is defined by $\{(y,x) \mid (x,y) \in R\}$.
  – If $R = \{(a,b), (a,c)\}$, then $R^\sim = \{(b,a), (c,a)\}$
  – $\rightarrow$ is the inverse of $<$ on the reals
• Note that $R^\sim \neq \sim R$
  – The complement of $<$ is $\geq$
  – But the inverse of $<$ is $>$
Set Operations

• Since relations are sets, set operations apply (just like relational algebra).

• The arity must be the same — indeed, the sets for the domain space and range space must be the same (just like in relational algebra).
Restriction & Extension

• Restriction decreases the domain space or range space.
  Example: Let < be the relation \{(1,2), (1,3), (2,3)\}. The restriction of the domain space to \{1\} restricts < to \{(1,2), (1,3)\}.

• Extensions increase the domain space or range space.
  Example: Let < be the relation \{(1,2), (1,3), (2,3)\}. The extension of both the domain space and range space to \{1,2,3,4\} extends < to \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}

• Extensions and restrictions need not change the relation.
  – Suppose \{M_1, M_2, M_3\} are three males and \{F_1, F_2, F_3\} are three females and married_to is \{(M_1,F_3), (M_2,F_2)\}.
  – Removing the unmarried male M_3 or the unmarried female F_1 does nothing to the relation, and adding the unmarried male M_4 or the unmarried female F_4 does nothing to the relation.
Composition

- Let $R:A \leftrightarrow B$ and $S:B \leftrightarrow C$ be two relations. The composition of $R$ and $S$, denoted by $R \circ S$, contains the pairs $(x,z)$ if and only if there is an intermediate object $y$ such that $(x,y)$ is in $R$ and $(y,z)$ is in $S$.

- Given $R:A \leftrightarrow B$ and $S:B \leftrightarrow C$, with $A = \{1,2,3,4\}$, $B = \{1,2,3,4,5\}$, $C = \{1,2,3,4\}$, and $R = \{(1,3), (4,2), (1,1)\}$ and $S = \{(3,4), (2,1), (4,2)\}$, we have:

$$R \circ S = \{(1,4), (4,1)\}$$
$$S \circ R = \{(3,2), (2,3), (2,1)\}$$  \hspace{1cm} \text{(i.e. not commutative)}
$$R \circ (S \circ R) = \{(1,2), (4,3), (4,1)\}$$
$$R \circ S \circ R = \{(1,2), (4,3), (4,1)\}$$  \hspace{1cm} \text{(i.e. is associative)}

Graphically:
Composition & Matrix Multiplication

• Relation matrices only contain 0’s and 1’s.
• For relational matrix multiplication, use $\land$ for $\ast$, $\lor$ for $+$, with $1=T$ and $0=F$.

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{pmatrix} \times \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}$$

$$R^O S_{(1,1)} = (1\land 0) \lor (0\land 1) \lor (1\land 0) \lor (0\land 0) \lor (0\land 0) = 0$$

$$R^O S = \{(1,4), (4,1)\}$$
Combined Operations

• All operations we have discussed can be combined so long as the compatibility requirements are met.
• Example:

\[ R = \{(1,1), (3,1), (1,2)\} \] 
\[ S = \{(1,1), (2,3)\} \] 
\[(R \cup S)^\circ R\] 

\[ = \{(1,1), (3,1), (1,2), (2,3)\}^\circ R \] 
\[ = \{(1,1), (1,3), (2,1), (3,2)\} \circ R \] 
\[ = \{(1,1), (1,2), (2,1), (2,2)\} \]
Properties of Binary Relations

• Properties of binary relations on a set $R: A \leftrightarrow A$ help us with lots of things — groupings, orderings, …

• Because there is only one set $A$:
  – matrices are square $|A| \times |A|$
  – graphs can be drawn with only one set of nodes

$$R = \{(1,3), (4,2), (3,3), (3,4)\}$$
Reflexivity

• Reflexive: $\forall x(xRx)$

• Irreflexive: $\forall x(xR\not x)$

  = is reflexive $\neq$ is irreflexive
  $\leq$ is reflexive $<$ is irreflexive

  “is in same major as” is reflexive
  “is sibling of” is irreflexive
  “loves” (unfortunately) is not reflexive, neither is it irreflexive
Symmetry

• Symmetric: \( \forall x \forall y (xRy \Rightarrow yRx) \)
• Antisymmetric: \( \forall x \forall y (xRy \land yRx \Rightarrow x = y) \)
• Asymmetric: \( \forall x \forall y (xRy \Rightarrow yRx) \)
  "sibling" is symmetric
  "brother_of" is not symmetric, in general, but symmetric if restricted to males
  \( \leq \) is antisymmetric (if \( a \leq b \) and \( b \leq a \), then \( a = b \))
  \(<\) is asymmetric and antisymmetric
  = is symmetric and antisymmetric
  \(\subseteq\) is antisymmetric
  "loves" (unfortunately) is not symmetric, neither is it antisymmetric nor asymmetric
Symmetry (continued…)

• symmetric: \[ \bullet \quad \text{always both ways} \]

• antisymmetric: \[ \text{but to self is ok} \]

• asymmetric: \[ \text{never both ways} \] (and never to self)

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Symmetric

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Antisymmetric (Asymmetric too, if no 1’s on the diagonal)

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No symmetry properties
Transitivity

- Transitive: \( \forall x \forall y \forall z (xRy \land yRz \Rightarrow xRz) \)
  - “taller” is transitive
  - \(<\) is transitive \(x<y<z\)
  - “ancestor” is transitive
  - “brother-in-law” is not transitive

- Transitive: “If I can get there in two, then I can get there in one.”

\[
\begin{align*}
\text{if} & \quad \text{and} & \quad \text{then} \\
x \quad & \quad y \quad & \quad z \\
(x \rightarrow y) \land (y \rightarrow z) & \Rightarrow (x \rightarrow z) \\
\end{align*}
\]

(for every \(x, y, z\))
Transitivity (continued…)

• Note: $xRy \land yRz$ corresponds to $xR^\circ Rz$, which is equivalent to $xR \times Rz = xR^2z$; thus we can define transitivity as:

$$\forall x \forall y \forall z(xR^2z \Rightarrow xRz)$$

“If I can get there in two, then I can get there in one.”

• For transitivity:
  – if reachable through an intermediate, then reachable directly is required.
  – if $(x,z) \in R^2$, then $(x,z) \in R$

$$\equiv (x,z) \in R^2 \Rightarrow (x,z) \in R$$

$$\equiv R^2 \subseteq R \text{ (recall our def. of subset: } x \in A \Rightarrow x \in B, \text{ then } A \subseteq B)$$
Transitivity (continued…)

Transitive

Not Transitive

Discussion #26