Discussion #17

Derivations
Topics

• Derivations — proofs in predicate calculus
• Inference Rules with Quantifiers
  – Laws
  – Universal/Existential Instantiation
  – Universal/Existential Generalization
• Unification
Laws as Inference Rules

• Equivalence laws can be used as inference rules.

• Show: if $\exists x P(x) \Rightarrow Q$ then $\forall x (P(x) \Rightarrow Q) \lor R$.

  1. $\exists x P(x) \Rightarrow Q$ premise
  2. $\neg \exists x P(x) \lor Q$ implication law
  3. $\forall x \neg P(x) \lor Q$ de Morgan’s law
  4. $\forall x (\neg P(x) \lor Q)$ distributive law
  5. $\forall x (P(x) \Rightarrow Q)$ implication law
  6. $\forall x (P(x) \Rightarrow Q) \lor R$ law of addition, $A \models A \lor B$

Note: the propositional laws of inference hold, as well (e.g. Step 6).
Universal Instantiation (UI)

\[ \forall x A \]

When A is true for every instantiation, it is certainly true for some particular instantiation.

Example: from \( \forall x \) Mortal(x), we can derive Mortal(Smith).
UI Example

Given the following two premises:

1. All intelligent students succeed
2. John is an intelligent student

Prove that John succeeds.

Proof:

1. $\forall x (\text{Intelligent}(x) \Rightarrow \text{Succeed}(x))$ premise
2. $\text{Intelligent}(\text{John})$ premise
3. $\text{Intelligent}(\text{John}) \Rightarrow \text{Succeed}(\text{John})$ 1, UI
4. $\text{Succeed}(\text{John})$ 2&3, modus ponens
Existential Instantiation (EI)

\[ \exists x A \]
\[ \underline{\exists x A} \quad \text{When } A \text{ is true for one or more} \]
\[ S^x_b A \quad \text{instantiations, we can let a} \]
\[ \quad \text{variable, say } b, \text{ designate any one} \]
\[ \quad \text{of the true instantiations.} \]

Example: from \( \exists x \text{ Mortal}(x) \), we can derive \( \text{Mortal}(b) \), where
\( b \) is a variable that stands for somebody who is mortal.

Caution: Because we don’t know which one(s) in the domain
make \( A \) hold, we must make sure \( b \) is a new variable — not
one already in use. For example, if we have \( \exists x \text{ Succeeds}(x) \)
and \( \exists x \text{ Fails}(x) \), then we cannot conclude \( \text{Succeeds}(b) \) and
\( \text{Fails}(b) \) — the first \( b \) is OK, but not the second.
EI Example

Given
1. Someone gets an A
2. Everyone who gets an A is happy

Prove that someone is happy

Translated to predicates:
1. \( \exists x \ \text{getsA}(x) \)
2. \( \forall x (\text{getsA}(x) \Rightarrow \text{happy}(x)) \)

Prove: \( \text{happy}(x) \)

Proof:
1. \( \exists x \ \text{getsA}(x) \) premise
2. \( \text{getsA}(b) \) 1, EI
3. \( \forall x (\text{getsA}(x) \Rightarrow \text{happy}(x)) \) premise
4. \( \text{getsA}(b) \Rightarrow \text{happy}(b) \) 3, UI
5. \( \text{happy}(b) \) 2&4, modus ponens
6. \( \text{happy}(x) \) 5, variant

Note: we could have used \( x \) in place of \( b \) to begin with. The \( x \)'s would not get mixed up because one is free and the others are bound. (This can be confusing and is not recommended.)
Existential Generalization (EG)

\[ S^x_t A \]

\[ \frac{S^x_t A}{\exists x A} \]

If we know \( A \) is true for some particular substitution \( t \), we know there exists at least one substitution for which \( A \) is true.
EG Example

Given
1. Someone gets an A
2. Everyone who gets an A is happy

Prove that there exists someone who is happy

Translated to predicates:
1. \( \exists x \text{ getsA}(x) \)
2. \( \forall x (\text{getsA}(x) \Rightarrow \text{happy}(x)) \)

Prove: \( \exists x \text{ happy}(x) \)

Proof:
1. \( \exists x \text{ getsA}(x) \) premise
2. \( \text{getsA}(b) \) 1, EI
3. \( \forall x (\text{getsA}(x) \Rightarrow \text{happy}(x)) \) premise
4. \( \text{getsA}(b) \Rightarrow \text{happy}(b) \) 3, UI
5. \( \text{happy}(b) \) 2&4, modus ponens
6. \( \exists x \text{ happy}(x) \) 5, EG
Universal Generalization (UG)

\[ \frac{A}{\forall x A} \quad \text{If } A \text{ holds with no restrictions on one of its variables } x, \text{ it must hold for all substitutions for } x. \]

Example: from \( \forall x (P(x) \Rightarrow Q(x)) \) and \( \forall x P(x) \), we can conclude \( P(a) \Rightarrow Q(a) \) and \( P(a) \) by UI and then \( Q(a) \) by modus ponens; and thus by UG we can conclude \( \forall x Q(x) \).

Caution: Any variable that represents a fixed element of the domain cannot be generalized by UG. Thus, the variable to be generalized must not be one that is introduced by EI and must not be one that is fixed in a premise.
Universal Generalization (UG) (continued…)

• Consider the following proof for “if ∀x(x>100 ⇒ x>10) and y>100, then y>10” which is correct.
  
  1. y>100  
  2. ∀x(x>100 ⇒ x>10)  
  3. y>100 ⇒ y>10  
  4. y>10  

• However, if we improperly use UG, we can conclude something that is clearly erroneous.

  1. y>100  
  2. ∀y(y>100)  
  3. …  

• It is not true that all numbers are greater than 100. In the original statement of what is to be proved, y is fixed (although unknown) and represents a particular element of the domain (namely, in this case, some number greater than 100).
Derivation Example

If $\forall x \forall y \forall z(\neg Q(x, y) \Rightarrow \neg P(z))$, $\forall w P(w)$
then $\forall x Q(x, 8)$.

1. $\forall x \forall y \forall z(\neg Q(x, y) \Rightarrow \neg P(z))$ premise
2. $\neg Q(a, b) \Rightarrow \neg P(c)$ 1, UI $S^x_a$, $S^y_b$, $S^z_c$
3. $P(c) \Rightarrow Q(a, b)$ 2, contrapositive
4. $\forall w P(w)$ premise
5. $P(c)$ 4, UI $S^w_c$
6. $Q(a, b)$ 3&5, modus ponens
7. $\forall x \forall y Q(x, y)$ 6, UG (Neither a nor b is free in a premise, nor were they introduced by EI.)
8. $\forall x Q(x, 8)$ 7, UI $S^y_8$
Derivation Example

If $\forall x \forall y \forall z (\neg Q(x, y) \Rightarrow \neg P(z))$, $\forall w P(w)$
then $\forall x Q(x, 8)$.

1. $\forall x \forall y \forall z (\neg Q(x, y) \Rightarrow \neg P(z))$ premise
2. $\neg Q(x, y) \Rightarrow \neg P(z)$ 1, UI $S^x_x$, $S^y_y$, $S^z_z$
3. $P(z) \Rightarrow Q(x, y)$ 2, contrapositive
4. $\forall w P(w)$ premise
5. $P(z)$ 4, UI $S^w_w$
6. $Q(x, y)$ 3&5, modus ponens
7. $\forall x \forall y Q(x, y)$ 6, UG (Neither x nor y is free in a premise, nor were they introduced by EI.)
8. $\forall x Q(x, 8)$ 7, UI $S^y_y$

Note: the names of the variables don’t matter, so long as we follow the rules properly.
Unification

The previous proof was basically about getting rid of universal quantifiers and then reintroducing them and about getting the variable names to match for modus ponens and for the conclusion. Since, we “play this game” over and over, we could (and should) simplify by doing two things:

1. Drop initial $\forall$’s, so long as we remember which variables are actually bound to these universal quantifiers so that we use them properly and which variables (if any) are fixed in the premises.
2. Use unification. Two expressions unify if there are legal instantiations that make them the same.
Simplified Derivation Example

If $\neg Q(x, y) \Rightarrow \neg P(z)$, $P(w)$ then $Q(x, 8)$.

1. $\neg Q(x, y) \Rightarrow \neg P(z)$ premise
2. $P(z) \Rightarrow Q(x, y)$ 1, contrapositive
3. $P(w)$ premise
4. $P(z)$ 3, unification with 2
5. $Q(x, y)$ 2&4, modus ponens
6. $Q(x, 8)$ 5, instantiation

Informally, we can see that this works:

If 2 & 3 are true for all substitutions, then we can always choose the w to be the same as z and be guaranteed that they will both be true (assuming the premises are true).

Note that the UI and UG are “hidden” within the unification.