Discussion #16

Validity & Equivalences
Topics

• Validity
  – Tautologies with Interpretations
  – Contradictions with Interpretations
• Logical Equivalences Involving Quantifiers
• Rectification
Validity

• An expression that is true for all interpretations is said to be valid. (A valid expression is also called a tautology.)

• An expression that is true for no interpretation is said to be contradictory. (A contradictory expression is also called a contradiction.)

• If A is valid, \( \neg A \) is contradictory.
  
  (a tautology)  (a contradiction)

• Examples:
  
  – \( P(x, y) \Rightarrow P(x, y) \equiv \neg P(x, y) \lor P(x, y) \) is valid
  
  – \( P(x, y) \land \neg P(x, y) \) is contradictory
Laws are Valid

• All laws are valid.
  – de Morgan’s: \( \neg(P(x) \land Q(y)) \equiv \neg P(x) \lor \neg Q(y) \)
  – Identity: \( P(x) \land T \equiv P(x) \)
• When we replace \( \equiv \) by \( \leftrightarrow \), the resulting expression is true for all interpretations.
  – de Morgan’s: \( \neg(P(x) \land Q(y)) \leftrightarrow \neg P(x) \lor \neg Q(y) \)
  – Identity: \( P(x) \land T \leftrightarrow P(x) \)
Equivalence with Variants

An expression with the variable names changed is called a variant. Proper variants are equivalent, i.e. it doesn’t matter what variable name is used.

Example: \( \forall x A \equiv \forall y S_y A \)

But, we must be careful

1. We must substitute only for the \( x \)’s bound by \( \forall x \).
2. Further, variables must not “clash.” Strong rule: \( y \) must not be in \( A \); weaker rule: no \( y \) in the scope of \( \forall x \) can be free in the scope of \( \forall x \), and no \( x \) bound by \( \forall x \) may be in the scope of a bound \( y \).

\[
\forall x (\exists y (P(y) \lor Q(x,z)) \land \exists x P(x))
\]

<table>
<thead>
<tr>
<th>Variable</th>
<th>Works</th>
<th>Doesn’t work</th>
</tr>
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<tbody>
<tr>
<td>( y )</td>
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Discussion #16
Equivalences Involving Quantifiers

1. \( \forall x A \equiv A \) if \( x \) not free in \( A \)

1d. \( \exists x A \equiv A \) if \( x \) not free in \( A \)

\begin{align*}
\forall x (\forall x P(x, z)) & \equiv \forall x P(x, z) \quad 1) \ x \text{ is already bound} \\
\forall x P(y, z) & \equiv P(y, z) \quad 2) \ \text{There are no } x \text{'s} \\
\forall x P(x, z) & \not\equiv P(x, z) \quad 3) \ x \text{ is free in } P(x, z)
\end{align*}

2. \( \forall x A \equiv \forall y S^x_y A \) if \( x \) does not “clash” with \( y \) in \( A \)

2d. \( \exists x A \equiv \exists y S^x_y A \) if \( x \) does not “clash” with \( y \) in \( A \)
Equivalences Involving Quantifiers (continued…)

3. \( \forall x A \equiv S_{t}^{x} A \land \forall x A \) for any term \( t \)

3d. \( \exists x A \equiv S_{t}^{x} A \lor \exists x A \) for any term \( t \)

1. When \( \forall x A \) is false, so is \( S_{t}^{x} A \land \forall x A \).

2. When \( \forall x A \) is true for all substitutions, \( S_{t}^{x} A \) is true, and hence \( S_{t}^{x} A \land \forall x A \) is true.

3. We are just “anding in” something that’s already there.

4. Dual argument for 3d (“oring in” something that’s already there).
Equivalences Involving Quantifiers
(continued…)

Distributive laws:

4.  \( \forall x(A \lor B) \equiv A \lor \forall xB \) if \( x \) not free in \( A \)
4d.  \( \exists x(A \land B) \equiv A \land \exists xB \) if \( x \) not free in \( A \)

\[ P \lor \forall xQ(x) \]
\[ \equiv P \lor (Q(x_1) \land Q(x_2) \land \ldots) \]
\[ \equiv (P \lor Q(x_1)) \land (P \lor Q(x_2)) \land \ldots \]
\[ \equiv \forall x(P \lor Q(x)) \]

Associative laws:

5.  \( \forall x(A \land B) \equiv \forall xA \land \forall xB \)
5d.  \( \exists x(A \lor B) \equiv \exists xA \lor \exists xB \)
Equivalences Involving Quantifiers (continued…)

Commutative laws:
6. $\forall x \forall y A \equiv \forall y \forall x A$
6d. $\exists x \exists y A \equiv \exists y \exists x A$

demorgan’s laws:
7. $\neg \exists x A \equiv \forall x \neg A$
7d. $\neg \forall x A \equiv \exists x \neg A$

$\neg \exists x P(x) \equiv \neg(P(x_1) \lor P(x_2) \lor \ldots)$
$\equiv \neg P(x_1) \land \neg P(x_2) \land \ldots$
$\equiv \forall x \neg P(x)$
Equivalence – Example

Show: $\exists x P(x) \Rightarrow Q \equiv \forall x (P(x) \Rightarrow Q)$

$\exists x P(x) \Rightarrow Q$

$\equiv \neg \exists x P(x) \lor Q$  \hspace{1cm} \text{implication law}$

$\equiv \forall x \neg P(x) \lor Q$  \hspace{1cm} \text{de Morgan’s law}$

$\equiv \forall x (\neg P(x) \lor Q)$  \hspace{1cm} \text{distributive law}$

$\equiv \forall x (P(x) \Rightarrow Q)$  \hspace{1cm} \text{implication law}$
Rectification

Standardizing variables apart, also called rectification — we can rename variables to make distinct variables have distinct names.

\((\forall x P(x, y) \iff \forall x Q(y, x)) \Rightarrow \exists y R(y, x)\)

\((\forall x P(x, y) \iff \forall z Q(y, z)) \Rightarrow \exists w R(w, v)\)
Universal Quantification of Free Variables in a Tautology

• Since $P(x, y) \implies P(x, y)$ is a tautology, it holds for every substitution of values for its variables for every interpretation.

• Thus, $P(x, y) \implies P(x, y)$

  $= \forall y (P(x, y) \implies P(x, y))$
  $= \forall x \forall y (P(x, y) \implies P(x, y))$

• Hence, we can drop the quantifiers for tautologies.