Discussion #13

Induction
(the process of deriving generalities from particulars)

Mathematical Induction
(deductive reasoning over the natural numbers)
Topics

• Proof by Induction
• Summary of Proof Techniques
Statements that Depend on a Natural Number

- Sometimes we wish to prove that a statement is true for all natural numbers $n \geq 0$ or for all natural numbers greater than some initial number. Examples:
  - Prove: The sum of the numbers 1 to $n$ is $n(n+1)/2$.
  - Prove: $n^2 > 2n + 1$, for $n \geq 3$.
  - Prove: If $T$ is a full binary tree, then the number of nodes at each level $n$ of $T$ is $2^n$.
  - Prove: If $E$ is an expression with $n$ occurrences of binary operators, then $E$ has $n+1$ operands.

- Each statement $S$ to be proved depends on a single number $n$.
  - We can write $S(n)$ to denote the statement that depends on $n$.
  - Thus, for the first statement above, $S(n)$ is “The sum of the numbers 1 through $n$ is $n(n+1)/2.””
Prove $S(n)$ for All $n \geq m$

• To prove a statement $S(n)$ for $n \geq m$, we must show that $S(m)$, $S(m+1)$, $S(m+2)$, … are all true.
  – Example: For $S(n) = \text{“The sum of the numbers 1 through } n \text{ is } n(n+1)/2\text{”}$ we must prove:
    • The sum of the numbers 1 to 1 is $1(1+1)/2$.
    • The sum of the numbers 1 to 2 is $2(2+1)/2$.
    • The sum of the numbers 1 to 3 is $3(3+1)/2$.
    • …
  – Unfortunately, we can never reach the end.

• The simple pattern of one after the next, however, lets us solve the problem.
If we can get started …

And then show for an arbitrary natural number \( k \) that this implication is a tautology …

We can conclude that \( S \) is true for the next natural number, \( k+1 \) (if it is true for \( k \)).

Then since \( k \) is chosen arbitrarily (works for any number), \( S \) must be true for \( k = 1, 2, 3, \ldots \) (in general, any number from the start number and beyond).
Induction Fundamentals by Example

Prove: S(n), e.g. Prove: “The sum of the numbers from 1 to n is n(n+1)/2.”

Basis: (i.e. this is how we get started)
S(1) holds.
i.e. The sum of the numbers from 1 to 1 is 1(1+1)/2 is a true statement.

The implication to be proved:
S(k) ⇒ S(k+1)
i.e. The sum of the numbers from 1 to k is k(k+1)/2
⇒ The sum of the numbers from 1 to k+1 is (k+1)((k+1)+1)/2

Induction: (i.e. this is the proof of the implication to be proved)
S(k) — premise (The premise is called the induction hypothesis.)
i.e. The sum of the numbers from 1 to k is k(k+1)/2.
If we add k+1 to k(k+1)/2, we have the sum of the numbers from 1 to k+1.
Thus, the sum of the numbers from 1 to k+1 is k+1+k(k+1)/2
= 2(k+1)/2 + k(k+1)/2 = (2(k+1)+k(k+1))/2 = (2k+2+k^2+k)/2
= (k^2+3k+2)/2 = (k+1)(k+2)/2 = (k+1)((k+1)+1)/2
i.e.S(k+1)
Prove: The sum of the numbers from 1 to $n$ is $n(n+1)/2$.

Basis: The sum of the numbers from 1 to 1 is $1(1+1)/2$.

Induction: By the induction hypothesis, the sum of the numbers from 1 to $k$ is $k(k+1)/2$. If we add $k+1$ to $k(k+1)/2$, we have the sum of the numbers from 1 to $k+1$. Thus, the sum of the numbers from 1 to $k+1$ is $k+1+k(k+1)/2$

$$= 2(k+1)/2 + k(k+1)/2 = (2(k+1)+k(k+1))/2 = (2k+2+k^2+k)/2$$

$$= (k^2+3k+2)/2 = (k+1)(k+2)/2 = (k+1)((k+1)+1)/2$$
Key Components

• Basis
  – Without the basis, we can “prove” erroneous conclusions.
  – Example: Prove? \( n+1 < n \).
    • We can prove: \( n+1 < n \Rightarrow (n+1)+1 < (n+1) \).
    • Proof: By the induction hypothesis, \( n+1 < n \). Adding 1 to both sides of an inequality retains the inequality. Thus, \( n+1+1 < n+1 \); hence \( (n+1)+1 < (n+1) \).

• Proof of the induction implication
  – Not just any implication from one natural number to the next holds.
  – Example: Prove? \( 2 – n > 0 \).
    • Basis: \( n=0 \), then \( 2 – 0 > 0 \).
    • We must now prove the induction implication: \( 2 – n > 0 \Rightarrow 2 – (n+1) > 0 \).
    • Induction: Assume \( 2 – n > 0 \). Thus, \( 2 – n – 1 > – 1 \) and hence \( 2 – (n + 1) > – 1 \). But now we’re stuck!
    • \( n = 1 \) is a counterexample since \( 2 – 1 > 0 \) is true while \( 2 – (1 + 1) > 0 \) is false.

• Use of the induction hypothesis
  – Left-hand-side of the implication to be proved, i.e. \( S(k) \) in \( S(k) \Rightarrow S(k+1) \)
  – A proof that does not use the induction hypothesis may be a valid proof, but it is not an induction proof.
Quadratic vs. Linear

Prove: \( n^2 > 2n + 1 \), for \( n \geq 3 \).

Basis: \( n = 3 \): \( 3^2 = 9 > 2(3) + 1 = 7 \)

Show: \( n^2 > 2n + 1 \Rightarrow (n+1)^2 > 2(n+1) + 1 \)

Induction hypothesis: \( n^2 > 2n + 1 \)

Note: \( n \geq 3 \) is a premise, which we can use as needed.

Induction:

\[
(n+1)^2 = n^2 + 2n + 1 \\
> 2n + 1 + 2n + 1 \quad \text{induction hypothesis} \\
= 2n + 2 + 2n \\
= 2(n+1) + 2n \\
> 2(n+1) + 1 \quad \text{since } 2n > 1 \text{ for } n \geq 3
\]
Prove: If T is a full binary tree, then the number of nodes at each level $n$ of T is $2^n$.

Full binary tree:

- $n=0$, $2^0 = 1$
- $n=1$, $2^1 = 2$
- $n=2$, $2^2 = 4$
- $n=3$, $2^3 = 8$
Full Binary Tree (continued…)

Prove: If $T$ is a full binary tree, then the number of nodes at each level $n$ of $T$ is $2^n$.

Basis: The number of nodes at level 0 is 1 (only the root) which is $2^0 = 1$.

Induction:

Let $T$ be a full binary tree, then, by the induction hypotheses, the number of nodes at level $k$ of $T$ is $2^k$. Since a full binary tree has 2 children for each node, there are $2 \times 2^k = 2^{k+1}$ nodes at level $k+1$. Thus, the number of nodes at level $k+1$ of $T$ is $2^{k+1}$.
Why Induction Works — by Example

• Consider the full binary tree proof just completed.
  – By our proof, we know that the basis is true: # of nodes at level 0 is $2^0$
  – Also by our proof, we know that the induction implication is true for any natural number $k$: # of nodes at level $k$ is $2^k$ $\Rightarrow$ # of nodes at level $k+1$ is $2^{k+1}$

• Thus, we can inductively unroll the proof by plugging in values.

1. # of nodes at level 0 is $2^0$ - basis
2. # of nodes at level 0 is $2^0 \Rightarrow$ # of nodes at level 1 is $2^1$ - ind. implic. with $k=0$
3. # of nodes at level 1 is $2^1$ - modus ponens, 1&2
4. # of nodes at level 1 is $2^1 \Rightarrow$ # of nodes at level 2 is $2^2$ - ind. implic. with $k=1$
5. # of nodes at level 2 is $2^2$ - modus ponens, 3&4
6. # of nodes at level 2 is $2^2 \Rightarrow$ # of nodes at level 3 is $2^3$ - ind. implic. with $k=2$
7. # of nodes at level 3 is $2^3$ - modus ponens, 5&6
8. # of nodes at level 3 is $2^3 \Rightarrow$ # of nodes at level 4 is $2^4$ - ind. implic. with $k=3$
9. # of nodes at level 4 is $2^4$ - modus ponens, 7&8
10. …

• Inductively, this goes on forever; and thus the statement holds for all natural numbers.
Since all of $S(1), \ldots, S(k)$ hold at this point, we can use any one or more or even all of $S(1), \ldots, S(k)$ to help prove $S(k+1)$. 
Number of Operands

Prove $S(n)$ : If $E$ is an expression with $n$ occurrences of binary operators, then $E$ has $n+1$ operands.

- e.g. $(2+3)\times((5+1)/3)$ has 4 operator occurrences and 5 operands

Basis: For $S(0)$, $E$ has 0 binary operators and 0+1 operands (just a single operand).

Induction: We must prove:

$S(0) \land \ldots \land S(k) \Rightarrow S(k+1)$. 
Number of Operands (continued...)  

Induction: Let $E$ be an expression with $k+1$ occurrences of binary operators. Then $E$ is of the form $E_1 \delta E_2$ where $E_1$ and $E_2$ are expressions and $\delta$ is the binary operator. Let $E_1$ have $k_1$ occurrences of binary operators and $E_2$ have $k_2$ occurrences of binary operators and thus $E$ has $k_1 + k_2 + 1 = k+1$ occurrences of binary operators. Now, $k_1 \leq k$ and $k_2 \leq k$. Hence, by the induction hypothesis, $E_1$ has $k_1+1$ operands and $E_2$ has $k_2+1$ operands. Thus $E$ has $k_1+1+k_2+1 = (k_1+k_2+1)+1 = (k+1)+1$ operands.

\[
\begin{align*}
E &= E_1 \delta E_2 \\
\text{k}_1 + \text{k}_2 + 1 &= k+1 \text{ operator occurrences}
\end{align*}
\]
Proof by Induction — Summary

• Observe that a proof by induction turns a proof into the form $P \implies Q$, a standard direct proof.

• Basis: Special case for the first, or first few, of the sequence, often $S(0)$.

• Induction:
  – Weak, 1\textsuperscript{st} Principle: $S(k) \implies S(k+1)$
  – Strong, 2\textsuperscript{nd} Principle: $S(0) \land \ldots \land S(k) \implies S(k+1)$

• The induction hypothesis, which must be used to prove the implication, is the lhs of the implication.
Proof Techniques — Summary

1. Exhaustive Truth Proof
   Show all cases.

1. Equivalence to Truth
   Reduce to T

1. Iff Proof
   Convert lhs to rhs
   \[ P \iff Q \equiv (P \Rightarrow Q) \land (Q \Rightarrow P) \]

1. Contrapositive Proof (special case of contradiction)
   \[ P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P \]

1. Contradiction (or indirect) Proof
   \[ P \equiv \neg P \Rightarrow F \]
   \[ P \Rightarrow Q \equiv P \land \neg Q \Rightarrow F \]
Proof Techniques — Summary
(continued …)

6. Conditional Proof
   \[ P \Rightarrow (Q \Rightarrow R) \equiv (P \land Q) \Rightarrow R \]

7. Case Analysis
   \[ P \equiv (Q \Rightarrow P) \land (\neg Q \Rightarrow P) \]

8. Proof by Induction
   Basis: \( S(0) \)
   Weak, 1\(^{\text{st}}\) Principle: \( S(k) \Rightarrow S(k+1) \)
   Strong, 2\(^{\text{nd}}\) Principle: \( S(0) \land \ldots \land S(k) \Rightarrow S(k+1) \)

9. Direct Proof
   \[ P \Rightarrow Q \]