Discussion #11

Logical Implications and Derivations
Topics

• Sound Arguments
• Derivations
• Proofs
  – Inference rules
  – Deduction
Sound Arguments

• A logical argument is sound if when all premises are true, the conclusion must also be true.

• Observe that if A represents the premises and C represents the conclusion, a logical argument is sound if $A \implies C$ is a tautology.

• A is an expression that can always be in CNF, in which case each term in the conjunction is called a premise, which by itself is true.
  
  – $A = P_1 \land P_2 \land \ldots \land P_n$ or just $P_1, P_2, \ldots P_n$.
  
  – Thus, what we wish to show is that $P_1 \land P_2 \land \ldots \land P_n \implies C$ is a tautology.

  – Note that if any one of the $P_i$’s is false, the implication is true; thus, we only have to worry about the case when all $P_i$’s are true. This idea is called the discharge rule because we can discharge the case in which the left-hand side is false.
Sound Argument

• Showing that $A \Rightarrow C$ is a tautology constitutes a sound logical argument.

• If $(A_1 \land A_2 \land \ldots \land A_n \Rightarrow C) = T$, and $A_i = T$ for all $i$, then $C$ must necessarily be true.

\[
\begin{array}{c|c|c|c}
A & C & A \Rightarrow C \\
T & ? & T \\
\end{array}
\]

C = T is the only possibility for the conclusion!
Deductions and Formal Proofs

• A *deduction* is a sequence of logic statements, each of which is known or assumed to be true.

• A *formal proof of a conclusion* \( C \) is a deduction that ends with \( C \).

• What is known or assumed to be true comes in three variations:
  – Premises that are assumed to be true
  – Any statement known to be true (e.g. \( 1+1 = 2 \))
  – Sound rules of inference that introduce logic statements that are true (assuming the statements they are based on are true.)
Example

Prove: if P, R, P ⇒ Q, ¬R ∨ S, then Q ∧ S.

1. P T? premise
2. P ⇒ Q T? premise
3. Q T 1&2, modus ponens (A, A⇒B |=B)
4. ¬R ∨ S T? premise
5. R ⇒ S T 4, implication (¬R ∨ S ≡ R ⇒ S)
6. R T? premise
7. S T 5&6, modus ponens
8. Q ∧ S T 3&7, law of combination (conjunction)
Notes...

• Thus, since we have a deduction (sequence of statements all of which are true) that ends with the conclusion, we have a formal proof.

• A formal proof guarantees that the original statement is a tautology.
  – If any premise is false, the statement is true.
  – If all premises are true, we are able to guarantee that the conclusion is true.
  – Thus, the statement is always true (i.e. is a tautology).

• Important note: A proof does not guarantee that the conclusion is true!
Valid Proof vs. Valid Conclusion

Consider the following proof:

If $2 > 3, \ 2 > 3 \Rightarrow 3 < 2$, then $3 < 2$.

1. $2 > 3$ premise
2. $2 > 3 \Rightarrow 3 < 2$ premise
3. $3 < 2$ 1&2, modus ponens

Is the proof valid?  Yes!  Sound argument

Is the conclusion valid?  No!

Hence, a valid proof does not guarantee a valid conclusion. What a valid proof does guarantee is that if the premises are all true, then the conclusion is true.
## Main Rules of Inference

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A, B \models A \land B )</td>
<td>Law of combination</td>
</tr>
<tr>
<td>( A \land B \models B )</td>
<td>Law of simplification</td>
</tr>
<tr>
<td>( A \land B \models A )</td>
<td>Variant of law of simplification</td>
</tr>
<tr>
<td>( A \models A \lor B )</td>
<td>Law of addition</td>
</tr>
<tr>
<td>( B \models A \lor B )</td>
<td>Variant of law of addition</td>
</tr>
<tr>
<td>( A, A \Rightarrow B \models B )</td>
<td>Modus ponens</td>
</tr>
<tr>
<td>( \neg B, A \Rightarrow B \models \neg A )</td>
<td>Modus tollens</td>
</tr>
<tr>
<td>( A \Rightarrow B, B \Rightarrow C \models A \Rightarrow C )</td>
<td>Hypothetical syllogism</td>
</tr>
<tr>
<td>( A \lor B, \neg A \models B )</td>
<td>Disjunctive syllogism</td>
</tr>
<tr>
<td>( A \lor B, \neg B \models A )</td>
<td>Variant of disjunctive syllogism</td>
</tr>
<tr>
<td>( A \Rightarrow B, \neg A \Rightarrow B \models B )</td>
<td>Law of cases</td>
</tr>
<tr>
<td>( A \iff B \models A \Rightarrow B )</td>
<td>Equivalence elimination</td>
</tr>
<tr>
<td>( A \iff B \models B \Rightarrow A )</td>
<td>Variant of equivalence elimination</td>
</tr>
<tr>
<td>( A \Rightarrow B, B \Rightarrow A \models A \iff B )</td>
<td>Equivalence introduction</td>
</tr>
<tr>
<td>( A, \neg A \models B )</td>
<td>Inconsistency law</td>
</tr>
</tbody>
</table>
Comments on Soundness of Rules

• Easy to show that the inference rules are sound.

\[
\begin{array}{c|ccc}
A & B & (A \land (A \implies B)) & \implies B \\
\hline
T & T & T & T \\
T & F & F & T \\
F & T & F & T \\
F & F & F & T \\
\end{array}
\]

Modus ponens

\[
A, A \implies B \models B
\]

• Some rules are obvious:
  
  − \( A, B \models A \land B \)
  
  − \( A \land B \models A \)
  
  − \( A \models A \lor B \)
Comments on Soundness of Rules: Some “Rules” are not Sound

• Easy to show that false, presumed, magical rules are not sound.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(B ∧ (A ⇒ B)) ⇒ A</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T T T T T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F F F T T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T T T F F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F T F T T</td>
</tr>
</tbody>
</table>

Abracadabra

B, A⇒B |= A?

• You cannot just use any implication as an inference rule. Only sound rules provide the guarantees we need.
Laws Used as Inference Rules

• Suppose \( A \equiv B \) is a law and \( A \) is \( T \), then since \( A \) is \( T \) and \( A \equiv B \), then \( B \) is \( T \).

• Thus, for \( A \equiv B \), we always have two rules of inference:

\[
\begin{align*}
\frac{A}{B} & \quad \frac{B}{A}
\end{align*}
\]
Example: Contrapositive

Since $P \implies Q \equiv \neg Q \implies \neg P$, we have the inference rules:

\[
\begin{align*}
\frac{P \implies Q}{\neg Q \implies \neg P} & \quad \frac{\neg Q \implies \neg P}{P \implies Q} \\
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>(P $\implies$ Q) $\iff$ (\neg Q $\implies$ \neg P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T F T F</td>
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<tr>
<td>T</td>
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Discussion #11
Reduction of Inference Laws

• Using laws, we can reduce the number of rules of inference.
  – Too many to remember
  – Too many to program! (Project)

• Example: modus tollens

\[
\neg B \\
A \Rightarrow B \\
\hline
\neg A
\]

alternative:

\[
A \Rightarrow B \\
\neg B \Rightarrow \neg A \\
\hline
\neg B \\
\neg A
\]
contrapositive

\[
\neg B \\
\neg A
\]
modus ponens
Reduction of Inference Laws

- Example: disjunctive syllogism

\[
\begin{align*}
A \lor B & \quad \text{alternative:} \quad A \lor B \\
\neg A & \quad \neg (\neg A) \lor B \quad \text{double negation} \\
B & \quad \neg A \Rightarrow B \quad P \Rightarrow Q \equiv \neg P \lor Q \\
\quad & \quad \neg A \quad \text{modus ponens}
\end{align*}
\]

- It turns out that modus ponens plus some simple laws are usually enough.

- It gets even simpler with resolution.