Discussion #9

Tautologies and Contradictions
Topics

- Definitions
- Sound reasoning with tautologies and contradictions
Definitions

- **Tautology** – a logical expression that is true for all variable assignments.
- **Contradiction** – a logical expression that is false for all variable assignments.
- **Contingent** – a logical expression that is neither a tautology nor a contradiction.
Since $P \lor \neg P$ is true for all variable assignments, it is a tautology.

<table>
<thead>
<tr>
<th>P</th>
<th>$\neg P$</th>
<th>$P \lor \neg P$</th>
<th>$\neg (P \lor \neg P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
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Since $P \lor \neg P$ is true for all variable assignments, it is a tautology.
Tautological Derivation by Substitution

Using schemas that are tautologies, we can get other tautologies by substituting expressions for schema variables.

- Since $A \lor \neg A$ is a tautology,
  
  so are $(P \equiv Q) \lor \neg (P \equiv Q)$
  
  and $(P \land Q \lor R) \lor \neg (P \land Q \lor R)$

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$A \Rightarrow B$</th>
<th>$A \land (A \Rightarrow B)$</th>
<th>$A \land (A \Rightarrow B) \Rightarrow B$</th>
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</thead>
<tbody>
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- Since $A \land (A \Rightarrow B) \Rightarrow B$ is a tautology,
  
  so is $(P \lor \neg Q) \land ((P \lor \neg Q) \Rightarrow R) \Rightarrow R$
Sound Reasoning

• A logical argument has the form:
  \[ A_1 \land A_2 \land \ldots \land A_n \Rightarrow B \]
  and is \textbf{sound} if when \( A_i = T \) for all \( i \), \( B = T \).
  (i.e. If the \textit{premises} are all true, then the \textit{conclusion} is also true.)

• This happens when \( A_1 \land A_2 \land \ldots \land A_n \Rightarrow B \) is a tautology.
Intuitive Basis for Sound Reasoning

If \((A_1 \land A_2 \land \ldots \land A_n \Rightarrow B)\) is a tautology, and \(A_i = T\) for all \(i\) then \(B\) must necessarily be true!

\[
\begin{array}{ccc}
A & B & A \Rightarrow B \\
T & ? & T \\
\end{array}
\]

\(B = T\) is the only possibility for the conclusion!
Modus Ponens

\[
\begin{array}{ccc}
A & B & A \Rightarrow B \\
T & ? & T \\
\end{array}
\]

\[
\begin{array}{cccc}
A & B & (A \Rightarrow B) & (A \Rightarrow B) \land A \\
T & T & T & T \\
T & F & F & F \\
F & T & T & F \\
F & F & T & T \\
\end{array}
\]

\[
\begin{array}{cccc}
A & B & (A \Rightarrow B) \land A \Rightarrow B \\
T & T & T & T \\
T & F & T & T \\
F & T & T & T \\
F & F & T & T \\
\end{array}
\]

Hence, modus ponens is sound.
## Disjunctive Syllogism

![Disjunctive Syllogism Diagram]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ∨ B</th>
<th>¬A</th>
<th>(A ∨ B) ∧ ¬A</th>
<th>(A ∨ B) ∧ ¬A ⇒ B</th>
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</thead>
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Hence, disjunctive syllogism is sound.