Chapter 4

BLENDING FUNCTIONS

We have just studied how the Bernstein polynomials serve very nicely as blending functions. We have noted that a degree n Bézier curve always begins at $\mathbf{P}_0$ and ends at $\mathbf{P}_n$. Also, the curve is always tangent to the control polygon at $\mathbf{P}_0$ and $\mathbf{P}_n$.

Other popular blending functions exist for defining curves. In fact, you can easily make up your own set of blending functions. And by following a few simple rules, you can actually create a new type of free-form curve which has desirable properties.

Consider a set of control points $\mathbf{P}_i$, $i = 0, ..., n$ and blending functions $f_i(t)$ which define the curve

$$\mathbf{P}(t) = \sum_{i=0}^{n} f_i(t)\mathbf{P}_i.$$

We can select our blending functions such that the curve has any or all of the following properties:

1. **Coordinate system independence**. This means that the curve will not change if the coordinate system is changed. In other words, imagine that the control points are drawn on a piece of paper and we move that piece of paper around so that the (x, y) coordinates of the control points change. It would be nice if the curve did not change relative to the control points. Actually, if we were to pick an arbitrary set of blending functions, the curve *would* change. In order to provide coordinate system independence, the blending functions must identically sum to one:

$$\sum_{i=0}^{n} f_i(t) \equiv 1.$$

2. **Convex hull property**. Curves which always lie within the convex hull of the control points are said to obey the convex hull property. The convex hull can be envisioned by imagining that each control point has a nail pounded into it. Imagine next that a rubber band is stretched so that it surrounds the group of nails, and then is allowed to collapse onto them. The polygon created by that rubber band is the convex hull. Bézier curves obey the convex hull property (Figure 4.1). This property exists in curves which are coordinate system independent and for which the blending functions are all non- negative:

$$\sum_{i=0}^{n} f_i(t) \equiv 1; \quad f_i(t) \geq 0, \quad 0 \leq t \leq 1, \quad i = 0, ..., n$$

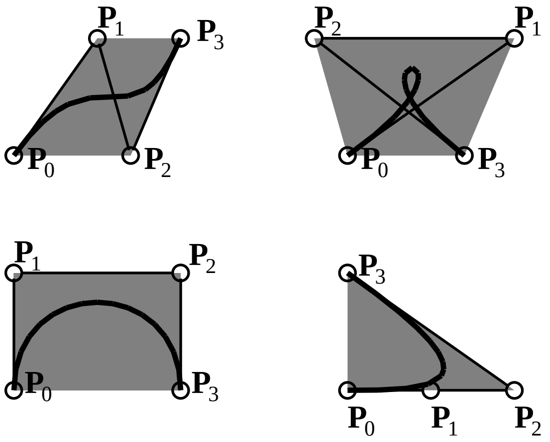

Figure 4.1: Convex Hull Property

3. **Symmetry**. Curves which are symmetric do not change if the control points are ordered in reverse sequence:

   $$\sum_{i=0}^{n} f_i(t)\mathbf{P}_i \equiv \sum_{i=0}^{n} f_i(1-t)\mathbf{P}_{n-i}.$$

   This holds if

   $$f_i(t) = f_{n-i}(1-t).$$

4. **Variation Diminishing Property**. This is a property which is obeyed by Bézier curves and B-spline curves. It states that if a given straight line intersects the curve in c number of points and the control polygon in p number of points, then it will always hold that

   $$c = p - 2j$$

   where j is zero or a positive integer. This has the practical interpretation that a curve which obeys the variation diminishing property will "wiggle" no more than the control polygon.

   The conditions under which a curve will obey the variation diminishing property are rather complicated. Suffice it to say that Bézier curves and B-spline curves obey this property, and most other curves do not.

5. **Linear Independence**. It is very desirable that the blending functions are linearly independent. If they are not linearly independent, then it is possible to express one blending function in terms of the other ones. This has the practical disadvantage that for certain control point arrangements, the curve collapses to a single point.

6. **Endpoint Interpolation** If a curve is to pass through the first and last control points, as in the case of Bézier curves, the following conditions must be met:

   $$f_0(0) = 1, \quad f_i(0) = 0, \; i = 1, ..., n$$

   $$f_n(1) = 1, \quad f_i(1) = 0, \; i = 0, ..., n-1.$$

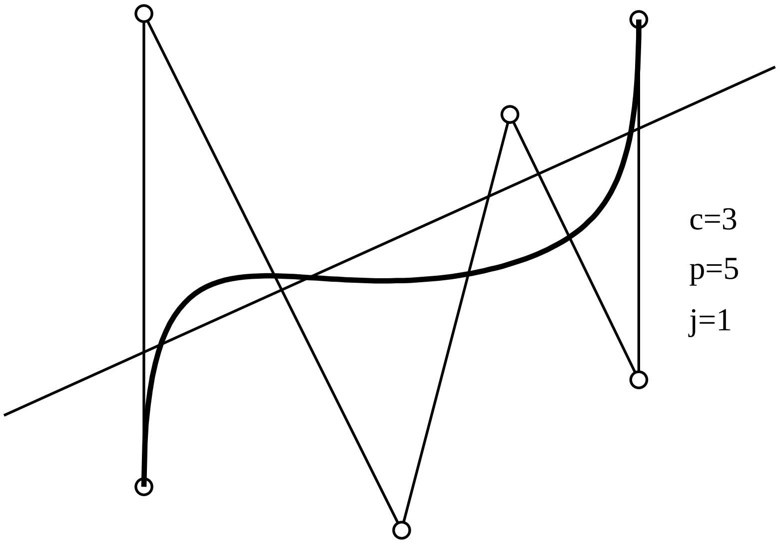

Figure 4.2: Variation Diminishing Property

The Bézier and B-spline curves are currently the most popular curve forms. Historically, other curve forms evolved independently at several different industrial sites, each faced with the common problem of making free-form curves accessible to designers without a mathematical background. In this section, we will review three such curves: Timmer's Parametric Cubic, Ball's Cubic, and the Overhauser curve. Each of these curves is coordinate system independent and symmetric, but only Ball's cubic obeys the convex hull property.

4.1 Timmer's Parametric Cubic

Timmer's Parametric Cubic (or PC) was created by Harry Timmer of McDonnell Douglas [54]. It was modeled after the Bézier curve. Timmer felt that he could improve upon the Bézier curve if he could make it follow the control polygon more tightly. This he did by forcing the curve to interpolate the endpoints of the control polygon and to be tangent to the control polygon at those points (just like Bézier curves) and in addition, he forced the curve to go through the midpoint of the line segment $\mathbf{P}_1$ - $\mathbf{P}_2$. The resulting blending functions are:

$$f_0(t) = (1-2t)(1-t)^2 = -2t^3 + 5t^2 - 4t + 1$$

$$f_1(t) = 4t(1-t)^2 = 4t^3 - 8t^2 + 4t$$

$$f_2(t) = 4t^2(1-t) = -4t^3 + 4t^2$$

$$f_3(t) = (2t-1)t^2 = 2t^3 - t^2$$

Figure 4.3 may mislead one into thinking that Timmer's curve is tangent to $\mathbf{P}_1 - \mathbf{P}_2$. This is not generally so (and is not exactly so even in this example).

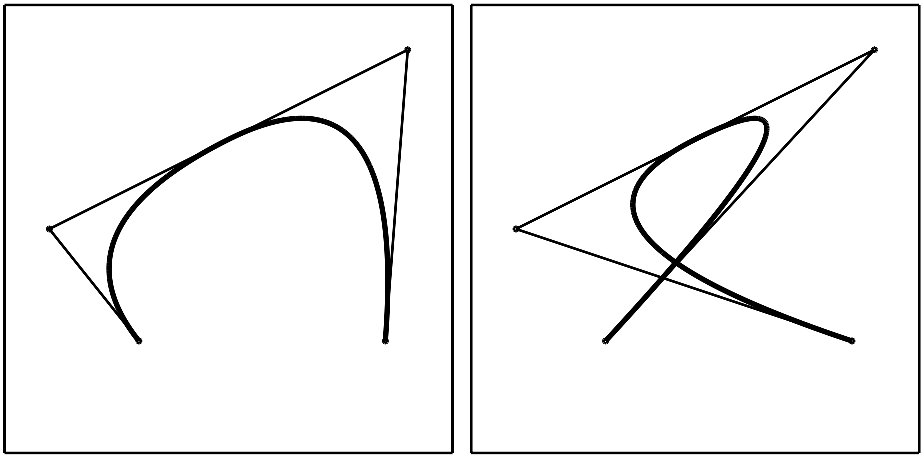

Figure 4.3: Timmer's PC

4.2 Ball's Rational Cubic

Alan Ball first published his cubic curve formulation in 1974 [1]. Ball worked for the British Aircraft Corporation, and his cubic curve form was used in BAC's in-house CAD system.

Like Timmer's PC curve, Ball's cubic can be considered a variant of the cubic Bézier curve. It's distinguishing feature is that it handles conic sections as a special case in a natural way. The blending functions for the non-rational case are:

$$f_0(t) = (1-t)^2$$

$$f_1(t) = 2t(1-t)^2$$

$$f_2(t) = 2t^2(1-t)$$

$$f_3(t) = t^2$$

Notice that if $\mathbf{P}_1 = \mathbf{P}_2$, then the curve becomes a quadratic Bézier curve:

$$\mathbf{P}_0(1-t)^2 + \mathbf{P}_1 2t(1-t)^2 + \mathbf{P}_1 2t^2(1-t) + \mathbf{P}_3 t^2$$

$$= \mathbf{P}_0(1-t)^2 + \mathbf{P}_1[2t(1-t)^2 + 2t^2(1-t)] + \mathbf{P}_3 t^2$$

$$= \mathbf{P}_0(1-t)^2 + \mathbf{P}_1 2t(1-t) + \mathbf{P}_3 t^2$$

4.3 Overhauser Curves

Overhauser curves were developed and used at Ford Motor Company [33]. They are also known as cubic Catmull-Rom splines [8].

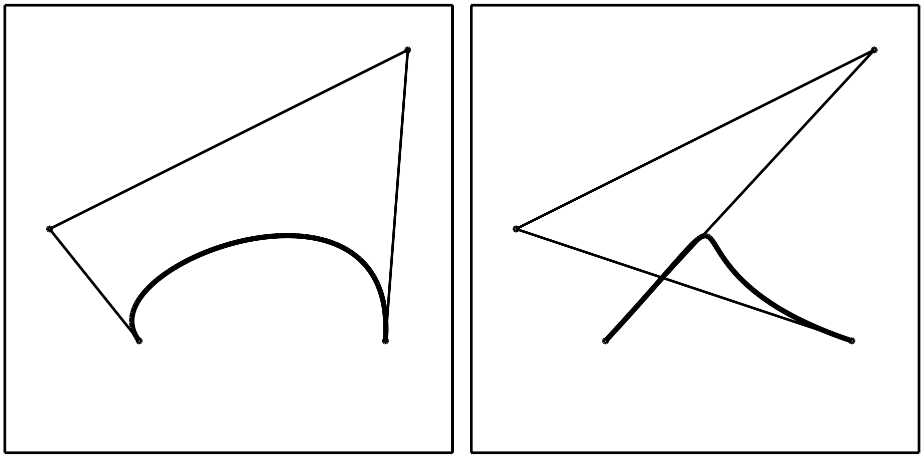

Figure 4.4: Ball's Cubic

One complaint of Bézier curves is that the curve does not interpolate all of the control points. Overhauser curves do interpolate all control points in a piecewise string of curve segments. A single Overhauser curve is defined with the following blending functions:

$$f_0(t) = -\frac{1}{2}t + t^2 - \frac{1}{2}t^3$$

$$f_1(t) = 1 - \frac{5}{2}t^2 + \frac{3}{2}t^3$$

$$f_2(t) = \frac{1}{2}t + 2t^2 - \frac{3}{2}t^3$$

$$f_3(t) = -\frac{1}{2}t^2 + \frac{1}{2}t^3$$

A single Overhauser curve segment interpolates $\mathbf{P}_1$ and $\mathbf{P}_2$. Furthermore, the slope of the curve at $\mathbf{P}_1$ is only a function of $\mathbf{P}_0$ and $\mathbf{P}_2$ and the slope at $\mathbf{P}_2$ is only a function of $\mathbf{P}_1$ and $\mathbf{P}_3$:

$$\mathbf{P}'(0) = \frac{1}{2}(\mathbf{P}_2 - \mathbf{P}_0)$$

$$\mathbf{P}'(1) = \frac{1}{2}(\mathbf{P}_3 - \mathbf{P}_1)$$

This means that a second curve segment will be tangent to the first curve segment if its first three control points are identical to the last three control points of the first curve. This is illustrated in Fig. 4.5.

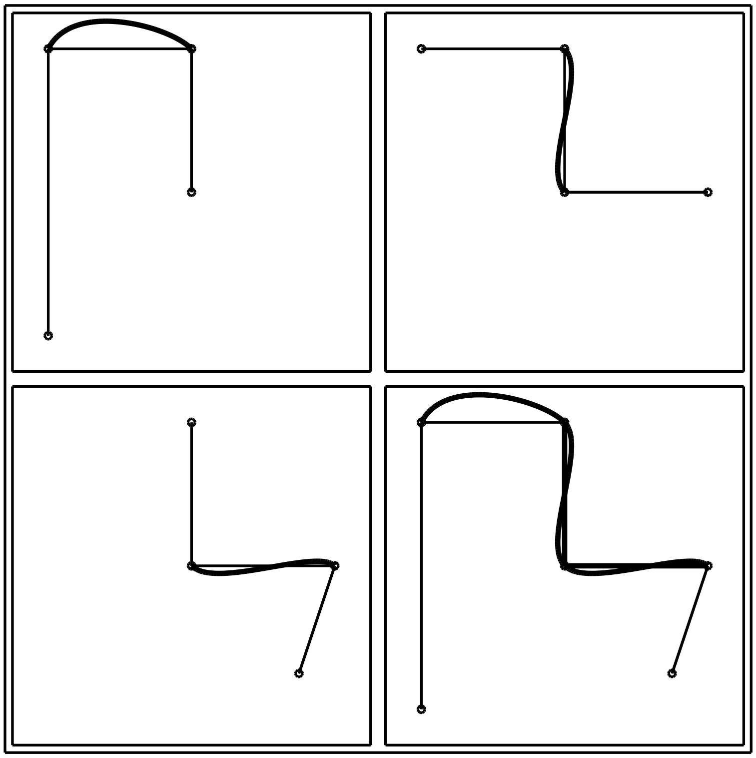

Figure 4.5: Overhauser curves