Chapter 8

ALGEBRAIC GEOMETRY FOR CAGD

The field of computer aided geometric design and graphics has drawn heavily from differential geometry and vector geometry, but is only beginning to access the tools of classical algebraic geometry. The word *classical* should be underscored, because much of *modern* algebraic geometry deals with abstractions which are far removed from the algorithmic nature of computer aided geometric design and graphics, whereas mathematicians of 50-150 years ago tended to write in less abstract terms.

The following pages present a collection of gleanings from algebraic geometry which hold practical value for computer aided geometric design and graphics. For example, the following three problems were first discussed a century ago but have lain dormant for decades:

1. Given a planar curve defined parametrically as $x = \frac{x(t)}{w(t)}$, $y = \frac{y(t)}{w(t)}$ where $x(t)$, $y(t)$, and $w(t)$ are polynomials, find an implicit equation $f(x, y) = 0$ which defines the same curve. This process of parametric to implicit conversion will be referred to as *implicitization.*

2. Given the (x, y) coordinates of a point which lies on a parametric curve $x = \frac{x(t)}{w(t)}$, $y = \frac{y(t)}{w(t)}$, find the parameter value t which corresponds to that point. This problem will be referred to as the *inversion* problem.

3. Compute the points of intersection of two parametric curves using the implicitization and inversion techniques.

These three problems are discussed in greatest detail in this tutorial. In addition, special characteristics of cubic parametric curves are presented, and several valuable properties of surfaces are discussed. Further information can be found in [41, 43].

Section 7.1 presents some preliminary terminology and theorems. Sections 7.2 through 7.5 discuss the implicitization and inversion of planar curves, and section 7.6 applies those tools to computing curve intersections. Section 7.7 discusses some special properties of parametric cubic curves and section 7.8 overviews surface implicitization.

8.1 Preliminaries

There are basically two ways that a planar curve can be defined: parametrically and implicitly. The parametric equation of a plane curve takes the form

$$x = \frac{x(t)}{w(t)} \qquad y = \frac{y(t)}{w(t)} \tag{20.1}$$

and the implicit equation of a curve is of the form

$$f(x, y) = 0. \tag{20.2}$$

In these notes, we restrict ourselves to the case where $x(t)$, $y(t)$, $w(t)$, and $f(x, y)$ are polynomials. Any curve that can be expressed parametrically as in equation 20.1 is referred to as a **rational** curve. In the classical algebraic geometry literature, a rational curve is sometimes called a **unicursal** curve, which means that it can be sketched without removing one's pencil from the paper. In computer aided geometric design, rational curves are referred as **rational polynomial parametric** curves or simply as **parametric** curves. The case where $w(t) \equiv 1$ is sometimes referred to as a **non-rational** parametric curve or an **integral** parametric curve. A curve which can be expressed in the form of equation 20.2 is known as a **planar algebraic** curve.

The parametric equation of a curve has the advantage of being able to quickly compute the (x, y) coordinates of points on the curve for plotting purposes. Also, it is simple to define a curve *segment* by restricting the parameter t to a finite range, for example $0 \le t \le 1$. On the other hand, the implicit equation of a curve enables one to easily determine whether a given point lies on the curve, or if not, which side of the curve it lies on.

A **rational surface** is one which can be expressed

$$x = \frac{x(s,t)}{w(s,t)} \qquad y = \frac{y(s,t)}{w(s,t)} \qquad z = \frac{z(s,t)}{w(s,t)} \tag{20.3}$$

where $x(s, t)$, $y(s, t)$, $z(s, t)$ and $w(s, t)$ are polynomials. Also, a surface which can be expressed by the equation

$$f(x, y, z) = 0 \tag{20.4}$$

where $f(x, y, z)$ is a polynomial is called an **algebraic surface.**

A **rational space curve** is one which can be expressed by the parametric equations

$$x = \frac{x(t)}{w(t)} \quad y = \frac{y(t)}{w(t)} \quad z = \frac{z(t)}{w(t)}. \tag{20.5}$$

The curve of intersection of two algebraic surfaces is an **algebraic space curve.**

Homogeneous coordinates are an important tool in algebraic geometry for describing points at infinity. A planar algebraic curve of degree n

$$f(x, y) = \sum_{i+j \le n} a_{ij} x^i y^j = 0$$

can be expressed in homogeneous form by introducing a homogenizing variable w as follows:

$$f(x, y, w) = \sum_{i+j+k=n} a_{ij} x^i y^j w^k = 0.$$

Note that, in the homogeneous equation, the degree of every term is n. For example, a circle whose equation is $(x - x_c)^2 + (y - y_c)^2 - r^2 = 0$ has the homogeneous representation

$$(X - x_c W)^2 + (Y - y_c W)^2 - r^2 W^2 = 0. \tag{20.6}$$

Any homogeneous triple (X, Y, W) corresponds to a point whose Cartesian coordinates (x, y) are $(\frac{X}{W}, \frac{Y}{W})$, which can be verified by dividing equation 20.6 by W^2. Notice that if $W = 0$, the Cartesian coordinates of the point are at infinity. For example, it is easy to verify using the homogeneous representation of equation 20.6 that *every* circle passes through the two points at infinity $(1, i, 0)$ and $(1, -i, 0)$. These special points are known as the circular points at infinity.

Homogeneous parameters are also used widely in algebraic geometry and can be useful for computer aided geometric design as well. For example, a unit circle at the origin can be expressed parametrically as

$$x = \frac{-t^2 + 1}{t^2 + 1} \qquad y = \frac{2t}{t^2 + 1}. \tag{20.7}$$

Alternately, the circle can be expressed in terms of homogeneous parameters (T, U) where $t = \frac{T}{U}$:

$$x = \frac{-T^2 + U^2}{T^2 + U^2} \qquad y = \frac{2TU}{T^2 + U^2}. \tag{20.8}$$

To plot the entire circle using equation 20.7, t would have to range from negative to positive infinity. An advantage to the homogeneous parameters is that the entire circle can be swept out with finite parameter values. This is done as follows:

- First Quadrant: $0 \le T \le 1$, $U = 1$
- Second Quadrant: $T = 1$, $1 \ge U \ge 0$
- Third Quadrant: $T = -1$, $0 \le U \le 1$
- Fourth Quadrant: $-1 \le T \le 0$, $U = 1$

The **fundamental theorem of algebra** states that a univariate polynomial of degree n has exactly n roots. That is, for $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ there exist exactly n values of t for which $f(t) = 0$, if we count complex roots and possible multiple roots. If the a_i are all real, then any complex roots must occur in conjugate pairs. In other words, if the complex number b+ci is a root of the real polynomial $f(t)$, then so also is b-ci.

Bezout's theorem states that a planar algebraic curve of degree n and a second planar algebraic curve of degree m intersect in exactly mn points, if we count complex intersections, intersections at infinity, and possible multiple intersections. If they intersect in more than that number of points, they intersect in infinitely many points. This happens if the two curves are identical, for example.

Consider the number of points at which two circles intersect. Circles are degree two curves, and thus we expect them to intersect in exactly four points. However, our practical experience with circles persuades us that they intersect in at most two points. Of course, two distinct circles *can* only intersect in at most two real points. However, we have seen from the homogeneous representation of a circle that all circles pass through the two circular points at infinity, and thus we account for the two missing intersection points.

Bezout's theorem also tells us that two surfaces of degree m and n respectively intersect in an algebraic space curve of degree mn. Also, a space curve of degree m intersects a surface of degree n in mn points. In fact, this provides us with a useful definition for the degree of a surface or space curve. The degree of a surface is the number of times it is intersected by a general line, and the degree of a space curve is the number of times it intersects a general plane. The phrase "number of intersections" must be modified by the phrase "properly counted", which means that complex, infinite, and multiple intersections require special consideration. A thorough explanation of these topics is well beyond the scope of this tutorial, and in fact their application to surfaces is not fully understood. However, it is of qualitative value for us to discuss the degrees of curves and surfaces, because the degree of a curve or surface is reflected in the degree of its implicit equation.

8.2 Implicitization

It was noted that there are basically two ways that a planar curve can be defined: parametrically ($x = x(t)/w(t)$, $y = y(t)/w(t)$) and implicitly ($f(x, y) = 0$).

Obviously, the parametric equation of a curve has the advantage of being able to quickly compute the (x, y) coordinates of several points on the curve for plotting purposes. Also, it is simple to define a curve *segment* by restricting the parameter t to a finite range, for example $0 \leq t \leq 1$. On the other hand, the implicit equation of a curve enables one to easily determine whether a given point lies on the curve, or if not, which side of the curve it lies on.

Given these two different equations for curves, it is natural to wonder if it is possible to convert between representations for a given curve. The answer is that it is *always* possible to find an implicit equation of a parametric curve, but a parametric equation can generally be found only for implicit curves of degree two or one. The process of finding the implicit equation of a curve which is expressed parametrically is referred to as *implicitization.* In section 7.5, we will discuss how this can be accomplished using an important algebraic tool, the *resultant,* and section 7.4 discusses resultants. For fun, section 7.3 illustrates how someone might tackle the implicitization problem before learning about resultants. Section 7.6 applies these ideas to the problem of intersecting two parametric curves.

8.3 Brute Force Implicization

Consider this simple example of parametric-to-implicit conversion: Given a line

$$x = t + 2 \qquad y = 3t + 1,$$

we can easily find an implicit equation which identically represents this line by solving for t as a function of x

$$t = x - 2$$

and substituting into the equation for y:

$$y = 3(x - 2) + 1$$

or $3x - y - 5 = 0$. Note that this implicit equation defines *precisely* the same curve as does the parametric equation. We can also identify two inversion equations (for finding the parameter value of a point on the line): $t = x - 2$ or $t = (y - 1)/3$.

This approach to implicitization also works for degree two parametric curves. Consider the parabola

$$x = t^2 + 1 \qquad y = t^2 + 2t - 2.$$

Again, we can solve for t as a function of x:

$$t \;=\; \pm\sqrt{x\;-\;1}$$

and substitute into the equation for y:

$$y \;=\; (\sqrt{x\;-\;1})^2 + -2\sqrt{x\;-\;1}\;-\;2.$$

We can isolate the radical and square both sides

$$(y\;-\;(x-1)\;+\;2)^2 \;=\; (\pm 2\sqrt{x-1})^2$$

to yield

$$x^2\;-\;2xy\;+\;y^2\;-\;10x\;+\;6y\;+\;13\;=\;0$$

which is the desired implicit equation. Again, this implicit equation defines exactly the same curve as does the parametric equation.

We run into trouble if we try to apply this implicitization technique to curves of degree higher than two. Note that the critical step is that we must be able to express t as a function of x. For cubic and quartic equations, this can be done, but the resulting expression is hopelessly complex. For curves of degree greater than four, it is simply not possible.

We cannot obtain an inversion equation for this parabola the way we did for the straight line. For example, suppose we want to find the parameter of the point $(5, -2)$ which we know to lie on the curve. The brute force approach would be to find the values of t which satisfy the equation

$$x \;=\; 5 \;=\; t^2\;+\;1$$

and then to compare them with the values of t which satisfy the equation

$$y \;=\; -2 \;=\; t^2\;+\;2t\;-\;2.$$

In the first case, we find t = -2 or 2, and in the second case, t = -2 or 0. The value of t which satisfies both equations is -2, which must therefore be the parameter value of the point (5,-2).

This unsuccessful attempt at implicitization and inversion motivates the following discussion of *resultants,* which will provide an elegant, general solution to the implicitization and inversion problems.

8.4 Resultants

Resultants address the question of whether two polynomials have a common root. Consider the two polynomials

$$f(t) \;=\; \sum_{i=0}^{n} a_i t^i \qquad g(t) \;=\; \sum_{i=0}^{n} b_i t^i$$

The resultant of $f(t)$ and $g(t)$, written $R(f,g)$, is an expression in terms of the coefficients a_i and b_i such that a common root of $f(t)$ and $g(t)$ exists if and only if $R(f,g) \;=\; 0$.

We illustrate by finding the resultant of two cubic polynomials

$$f(t) \;=\; a_3t^3\;+\;a_2t^2\;+\;a_1t\;+\;a_0 \quad g(t) \;=\; b_3t^3\;+\;b_2t^2\;+\;b_1t\;+\;b_0.$$

In other words, we want to determine whether there exists a value α such that $f(\alpha) = g(\alpha) = 0$ – without having to actually find all roots of both polynomials and comparing. We begin by forming three auxiliary polynomials $h_1(t)$, $h_2(t)$ and $h_3(t)$ as follows:

$$\begin{aligned} h_1(t) &= a_3 g(t) - b_3 f(t) \\ &= (a_3 b_2) t^2 + (a_3 b_1) t + (a_3 b_0) \end{aligned}$$

where $(a_i b_j) \equiv (a_i b_j - a_j b_i)$ and

$$\begin{aligned} h_2(t) &= (a_3 t + a_2) g(t) - (b_3 t + b_2) f(t) \\ &= (a_3 b_1) t^2 + [(a_3 b_0) + (a_2 b_1)] t + (a_2 b_0) \end{aligned}$$

$$\begin{aligned} h_3(t) &= (a_3 t^2 + a_2 t + a_1) g(t) - (b_3 t^2 + b_2 t + b_1) f(t) \\ &= (a_3 b_0) t^2 + (a_2 b_0) t + (a_1 b_0) \end{aligned}$$

Note that if there exists a value α such that $f(\alpha) = g(\alpha) = 0$, then $h_1(\alpha) = h_2(\alpha) = h_3(\alpha) = 0$. We can therefore say that $f(t)$ and $g(t)$ have a common root if and only if the set of equations

$$\begin{bmatrix} (a_3 b_2) & (a_3 b_1) & (a_3 b_0) \\ (a_3 b_1) & (a_3 b_0) + (a_2 b_1) & (a_2 b_0) \\ (a_3 b_0) & (a_2 b_0) & (a_1 b_0) \end{bmatrix} \begin{Bmatrix} t^2 \\ t \\ 1 \end{Bmatrix} = 0$$

has a solution. [1] However, we know from linear algebra that this set of homogeneous linear equations can have a solution if and only if

$$\begin{vmatrix} (a_3 b_2) & (a_3 b_1) & (a_3 b_0) \\ (a_3 b_1) & (a_3 b_0) + (a_2 b_1) & (a_2 b_0) \\ (a_3 b_0) & (a_2 b_0) & (a_1 b_0) \end{vmatrix} = 0$$

and therefore,

$$R(f,g) = \begin{vmatrix} (a_3 b_2) & (a_3 b_1) & (a_3 b_0) \\ (a_3 b_1) & (a_3 b_0) + (a_2 b_1) & (a_2 b_0) \\ (a_3 b_0) & (a_2 b_0) & (a_1 b_0) \end{vmatrix}$$

This same approach can be used to construct the resultant of polynomials of any degree.

Let's try this resultant on a couple of examples. First, let $f(t) = t^3 - 2t^2 + 3t + 1$ and $g(t) = 2t^3 + 3t^2 - t + 4$. For this case,

$$R(f,g) = \begin{vmatrix} 7 & -7 & 2 \\ -7 & -5 & -11 \\ 2 & -11 & 13 \end{vmatrix} = -1611$$

We aren't so much interested in the actual numerical value of the resultant, just whether it equals zero or not. In this case, $R(f,g) = -1611 \neq 0$, so we conclude that $f(t)$ and $g(t)$ do *not* have a common root.

Consider next the pair of polynomials $f(t) = t^3 - t^2 - 11t - 4$ and $g(t) = 2t^3$ - $7t^2$ - $5t$ + 4. In this case,

$$R(f,g) = \begin{vmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \\ 12 & -32 & -64 \end{vmatrix} = 0$$

[1] Actually, we have only shown that this is a necessary condition. The proof that it is also sufficient can be found in [16].

Since $R(f,g) = 0$, $f(t)$ and $g(t)$ *do* have a common root. Note that the resultant simply determines the existence or non-existence of a common root, but it does not directly reveal the value of a common root, if one exists. In fact, if the resultant is zero, there may actually be several common roots. The next section discusses how to compute the common root(s).

8.5 Determining the Common Root

We present two basic approaches to finding the common root of two polynomials: by solving a set of linear equations, or by using Euclid's algorithm.

Linear Equation Approach Our intuitive development of the resultant of two cubic polynomials led us to a set of three linear equations in three "unknowns": t^2, t and 1. In general, we could create the resultant of two degree n polynomials $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, $g(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0$, as the determinant of the coefficient matrix of n homogeneous linear equations:

$$\begin{bmatrix} (a_n b_{n-1}) & . & . & . & (a_n b_0) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ (a_n b_0) & . & . & . & (a_1 b_0) \end{bmatrix} \begin{Bmatrix} t^{n-1} \\ t^{n-2} \\ . \\ . \\ t \\ 1 \end{Bmatrix} = 0$$

It may be a bit confusing at first to view this as a set of homogeneous *linear* equations, since the unknowns are all powers of t. Let us temporarily rename our unknowns:

$$\begin{bmatrix} (a_n b_{n-1}) & . & . & . & (a_n b_0) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ (a_n b_0) & . & . & . & (a_1 b_0) \end{bmatrix} \begin{Bmatrix} X_{n-1} \\ X_{n-2} \\ . \\ . \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

where $X_i = t^i$. After solving for the X_i, the common root of $f(t)$ and $g(t)$ can be obtained as $t = X_{i+1}/X_i$.

Cramer's Rule There are several well known methods for solving for the X_i. One way is to apply Cramer's rule. A non-trivial solution exists (that is, a solution other than $X_0 = X_1 = ... = X_{n-1} = 0$) only if the determinant of the matrix is zero. But, that implies that the n equations are linearly dependent and we can discard one of them without losing any information. We discard the *last* equation, and can then solve for $n-1$ homogeneous equations in n homogeneous unknowns using Cramer's rule. It turns out that occasionally we run into trouble if we discard an equation other than the last one.We illustrate Cramer's rule for the case $f(t) = t^3 - t^2 - 11t - 4$ and $g(t) = 2t^3 - 7t^2 - 5t + 4$. Recall that this is the pair for which we earlier found that $R(f,g) = 0$. We have the set of equations

$$\begin{bmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \\ 12 & -32 & -64 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

Discarding the last equation, we obtain

$$\begin{bmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

from which we find the common root using Cramer's rule:

$$t = \frac{X_1}{X_0} = -\frac{\begin{vmatrix} -5 & 12 \\ 17 & -32 \end{vmatrix}}{\begin{vmatrix} -5 & 17 \\ 17 & -60 \end{vmatrix}} = 4$$

Gauss Elimination A numerically superior algorithm for solving this set of equations is to perform Gauss elimination. Two other advantages of Gauss elimination are that it can be used to determine whether the determinant is zero to begin with, and also it reveals *how many* common roots there are. We will illustrate this approach with three examples, using integer preserving Gauss elimination. We choose the integer preserving Gauss elimination because then the lower right hand element of the upper triangular matrix is the value of the determinant of the matrix.

Example 1

Our first example is one we considered earlier: $f(t) = t^3 - 2t^2 + 3t + 1$ and $g(t) = 2t^3 + 3t^2 - t + 4$. We set up the following set of linear equations, and triangularize the matrix using integer preserving Gauss elimination:

$$\begin{bmatrix} 7 & -7 & 2 \\ -7 & -5 & -11 \\ 2 & -11 & 13 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} =$$

$$\begin{bmatrix} 7 & -7 & 2 \\ 0 & -84 & 0 \\ 0 & 0 & -1611 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

We observe that the only solution to this set of equations is $X_2 = X_1 = X_0 = 0$, and conclude that $f(t)$ and $g(t)$ do not have a common root. Note that the lower right element -1611 is the determinant of the original matrix, or the resultant.

Example 2

We next examine the pair of polynomials $f(t) = t^3 - t^2 - 11t - 4$ and $g(t) = 2t^3 - 7t^2 - 5t + 4$. In this case, we have

$$\begin{bmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \\ 12 & -32 & -64 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} =$$

$$\begin{bmatrix} -5 & 17 & 12 \\ 0 & 11 & -44 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

Again, the bottom right element is the value of the determinant, which verifies that the resultant is zero. It is now simple to compute the solution: $X_1 = 4X_0$, $X_2 = 4X_1$. Since $X_i = t^i$, the common root is $t = 4$.

Example 3

For our final example we analyze the polynomials $f(t) = t^3 - 6t^2 + 11t - 6$ and $g(t) = t^3 - 7t^2 + 14t - 8$. Our linear equations now are:

$$\begin{bmatrix} -1 & 3 & -2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

In this case, not only is the resultant zero, but the matrix is rank 1. This means that there are *two* common roots, and they can be found as the solution to the quadratic equation $-t^2 + 3t - 2$, which is $t = 1$ and $t = 2$. Another way of saying this is that $-t^2 + 3t - 2$ is the *Greatest Common Divisor* of $f(t)$ and $g(t)$.

Euclid's GCD Algorithm An alternative approach to finding the common root(s) of two polynomials is to use Euclid's algorithm. This ancient algorithm can be used to find the *Greatest Common Divisor* of two integers, or two polynomials. Our presentation consists of a series of examples. This algorithm works beautifully in exact integer arithmetic, but we have experienced numerical instability in floating point.

Integer Example

We illustrate first on a pair of integers: 42 and 30. For the first step, we assign the larger to be the numerator, and the other to be the denominator:

Step 1: $\frac{42}{30} = 1$ remainder 12.

We now take the remainder of the first step and divide it into the denominator of the first step:

Step 2: $\frac{30}{12} = 2$ remainder 6.

We continue dividing the remainder of the preceding step into the denominator of the preceding step, until we obtain a zero remainder. This happens to occur in the third step for this problem:

Step 3: $\frac{12}{6} = 2$ exactly.

According to Euclid's algorithm, the second to last remainder is the GCD. In this case, the second to last remainder is 6, which is clearly the largest integer that evenly divides 30 and 42.

Polynomial Example 1

We illustrate how Euclid's algorithm works for polynomials by using the same three examples we used in the previous section. For the polynomials $f(t) = t^3 - 2t^2 + 3t + 1$ and $g(t) = 2t^3 + 3t^2 - t + 4$, we have:

- Step 1: $\frac{2t^3 + 3t^2 - t + 4}{t^3 - 2t^2 + 3t + 1} = 2$ remainder $7t^2 - 7t + 2$.
- Step 2: $\frac{t^3 - 2t^2 + 3t + 1}{7t^2 - 7t + 2} = \frac{t - 1}{7}$ remainder $\frac{12t+9}{7}$
- Step 3: $\frac{7t^2 - 7t + 2}{(12t + 9)/7} = \frac{196t - 343}{48}$ remainder $\frac{1253}{16}$
- Step 4: $\frac{(12t + 9)/7}{1253/16} = \frac{192t}{8771} + \frac{144}{8771}$ remainder 0.

In this case, the GCD is $\frac{1253}{16}$, which is merely a constant, and so $f(t)$ and $g(t)$ do not have a common root.

Polynomial Example 2

We next analyze the polynomials $f(t) = t^3 - t^2 - 11t - 4$ and $g(t) = 2t^3 - 7t^2 - 5t + 4$:

- Step 1: $\frac{2t^3 - 7t^2 - 5t + 4}{t^3 - t^2 - 11t - 4} = 2$ remainder $-5t^2 + 17t + 12$.
- Step 2: $\frac{t^3 - t^2 - 11t - 4}{-5t^2 + 17t + 12} = -5t - \frac{12}{25}$ remainder $\frac{-11t + 44}{25}$
- Step 3: $\frac{-5t^2 + 17t + 12}{(-11t + 44)/25} = 125t + \frac{75}{11}$ remainder 0.

In this case, the GCD is $\frac{-11t + 44}{25}$, and the common root is $t = 4$.

Polynomial Example 3

Finally, consider $f(t) = t^3 - 6t^2 + 11t - 6$ and $g(t) = t^3 - 7t^2 + 14t - 8$:

- Step 1: $\frac{t^3 - 6t^2 + 11t - 6}{t^3 - 7t^2 + 14t - 8} = 1$ remainder $t^2 - 3t + 2$
- Step 2: $\frac{t^3 - 7t^2 + 14t - 8}{t^2 - 3t + 2} = t - 4$ remainder 0.

The GCD is $t^2 - 3t + 2$, and the common roots are the roots of the equation $t^2 - 3t + 2 = 0$ which are $t = 1$ and $t = 2$.

You may have realized that there is a close connection between Euclid's algorithm and resultants, and obviously Euclid's algorithm does everything for us that resultants do.

We are now prepared to apply these tools to the problems of implicitizing and inverting curves.

8.6 Implicitization and Inversion

We discussed in the previous section a tool for determining whether two polynomials have a common root. We want to apply that tool to converting the parametric equation of a curve given by $x = \frac{x(t)}{w(t)}$, $y = \frac{y(t)}{w(t)}$ into an implicit equation of the form $f(x, y) = 0$. We proceed by forming two auxiliary polynomials:

$$p(x, t) = w(t)x - x(t) \qquad q(y, t) = w(t)y - y(t)$$

Note that $p(x, t) = q(y, t) = 0$ only for values of x, y, and t which satisfy the relationships $x = \frac{x(t)}{w(t)}$ and $y = \frac{y(t)}{w(t)}$. View $p(x, t)$ as a polynomial in t whose coefficients are linear in x, and view $q(y, t)$ as a polynomial in t whose coefficients are linear in y. If

$$x(t) = \sum_{i=0}^{n} a_i t^i, \quad y(t) = \sum_{i=0}^{n} b_i t^i, \quad w(t) = \sum_{i=0}^{n} d_i t^i$$

then

$$p(x,t) = (d_n x - a_n)t^n + (d_{n-1}x - a_{n-1})t^{n-1} + \cdots$$
$$+ (d_1 x - a_1)t + (d_0 x - a_0)$$
$$q(y,t) = (d_n y - b_n)t^n + (d_{n-1}y - b_{n-1})t^{n-1} + \cdots$$
$$+ (d_1 y - b_1)t + (d_0 y - b_0)$$

If we now compute the resultant of $p(x,t)$ and $q(y,t)$, we do not arrive at a numerical value, but rather a *polynomial* in x and y which we shall call $f(x,y)$. Clearly, any (x,y) pair for which $f(x,y) = 0$, causes the resultant of p and q to be zero. But, if the resultant is zero, then we know that there exists a value of t for which $p(x,t) = q(y,t) = 0$. In other words, all (x,y) for which $f(x,y) = 0$ lie on the parametric curve and therefore $f(x,y) = 0$ is the implicit equation of that curve. This should be clarified by the following examples.

Implicitization Example 1

Let's begin by applying this technique to the parabola we implicitized earlier using a brute force method:

$$x = t^2 + 1 \qquad y = t^2 + 2t - 2.$$

We begin by forming $p(x,t) = -t^2 + (x-1)$ and $q(y,t) = -t^2 - 2t + (y+2)$. The resultant of two quadratic polynomials $a_2t^2 + a_1t + a_0$ and $b_2t^2 + b_1t + b_0$ is

$$\begin{vmatrix} (a_2b_1) & (a_2b_0) \\ (a_2b_0) & (a_1b_0) \end{vmatrix}$$

and so the resultant of $p(x,t)$ and $q(y,t)$ is

$$R(p,q) = \begin{vmatrix} 2 & x-y-3 \\ x-y-3 & 2x-2 \end{vmatrix} =$$

$$-x^2 + 2xy - y^2 + 10x - 6y - 13$$

which is the implicit equation we arrived at earlier.

We can write an inversion equation for this curve – something which eluded us in our ad hoc approach:

$$\begin{bmatrix} 2 & x-y-3 \\ x-y-3 & 2x-2 \end{bmatrix} \begin{Bmatrix} t \\ 1 \end{Bmatrix} = 0$$

From which $t = \frac{-x + y + 3}{2}$ or $t = \frac{-2x + 2}{x-y-3}$.

Implicitization Example 2

We now implicitize the cubic curve for which

$$x = \frac{2t^3 - 18t^2 + 18t + 4}{-3t^2 + 3t + 1}$$

$$y = \frac{39t^3 - 69t^2 + 33t + 1}{-3t^2 + 3t + 1}$$

We begin by forming $p(x,t)$ and $q(y,t)$:

$$p(x,t) \;=\; -\,2t^3 \;+\; (-3x \;+\; 18)t^2 \;+\; (3x \;-\; 18)t \;+\; (x \;-\; 4)$$

$$q(y,t) \;=\; -\,39t^3 \;+\; (-3y \;+\; 69)t^2 \;+\; (3y \;-\; 33)t \;+\; (y-1)$$

Recalling from section 20.3 that the resultant of two cubic polynomials $a_3t^3 \;+\; a_2t^2 \;+\; a_1t \;+\; a_0$ and $a_3t^3 \;+\; a_2t^2 \;+\; a_1t \;+\; a_0$ is

$$\begin{vmatrix} (a_3b_2) & (a_3b_1) & (a_3b_0) \\ (a_3b_1) & (a_3b_0)+(a_2b_1) & (a_2b_0) \\ (a_3b_0) & (a_2b_0) & (a_1b_0) \end{vmatrix},$$

we have

$$R(p,q) \;=\; f(x,y) \;=$$

$$\begin{vmatrix} -117x \;+\; 69y \;+\; 564 & 117x \;-\; 6y \;-\; 636 & 39x \;-\; 2y \;-\; 154 \\ 117x \;-\; 6y \;-\; 636 & -69x \;-\; 2y \;+\; 494 & -66x \;+\; 6y \;+\; 258 \\ 39x \;-\; 2y \;-\; 154 & -66x \;-\; 2y \;+\; 258 & 30x \;-\; 6y \;-\; 114 \end{vmatrix}.$$

We can expand the determinant to get

$$f(x,y) \;=\; -\,156195x^3 \;+\; 60426x^2y \;-\; 7056xy^2 \;+\; 224y^3 \;+$$

$$2188998x^2 \;-\; 562500xy \;+\; 33168y^2 \;-\; 10175796x \;+\; 1322088y \;+$$

$$15631624$$

We can obtain an inversion equation using Cramer's rule:

$$t \;=\; \frac{t^2}{t} \;=\; -\,\frac{\begin{vmatrix} (117x \;-\; 6y \;-\; 636) & (39x \;-\; 2y \;-\; 154) \\ (-69x \;-\; 2y \;+\; 494) & (-66x \;+\; 6y \;+\; 258) \end{vmatrix}}{\begin{vmatrix} (-117x \;+\; 69y \;+\; 564) & (39x \;-\; 2y \;-\; 154) \\ (117x \;-\; 6y \;-\; 636) & (-66x \;+\; 6y \;+\; 258) \end{vmatrix}}$$

Alternately, we could use Gauss elimination to compute the parameter of a point on the curve. Cramer's rule has the appeal that it actually generates an *equation.*

We have intentionally carried out all computations in exact integer arithmetic to emphasize the rational, non-iterative nature of implicitization and inversion. Since the coefficients of the implicit equation are obtained from the coefficients of the parametric equations using only multiplication, addition and subtraction, it is possible to obtain an implicit equation which *precisely* defines the same point set as is defined by the parametric equations.

8.7 Curve-Curve Intersections

Given one curve defined by the implicit equation $f(x,y) \;=\; 0$ and a second curve defined by the parametric equations $x \;=\; x(t),\; y \;=\; y(t)$, we replace all occurrences of x in the implicit equation by $x(t)$, and replace all occurrences of y in the implicit equation by $y(t)$. These substitutions create a polynomial $f(x(t),y(t)) \;=\; g(t)$ whose roots are the parameter values of the points of intersection. Of course, if we start off with two parametric curves, we can first implicitize one of them.

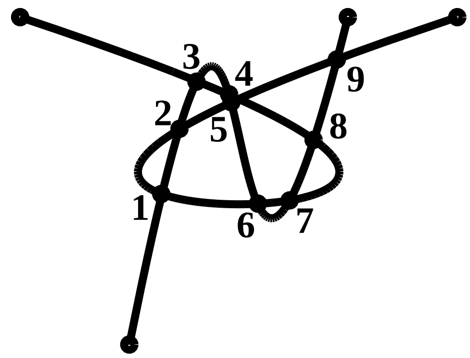

Figure 8.1: Two cubic curves intersecting nine times

We illustrate this process by intersecting the curve

$$x = \frac{2t_1^3 - 18t_1^2 + 18t_1 + 4}{-3t_1^2 + 3t_1 + 1}$$

$$y = \frac{39t_1^3 - 69t_1^2 + 33t_1 + 1}{-3t_1^2 + 3t_1 + 1}$$

with the curve

$$x = \frac{-52t_2^3 + 63t_2^2 - 15t_2 + 7}{-37t_2^2 + 3t_2 + 1}$$

$$y = \frac{4}{-37t_2^2 + 3t_2 + 1}$$

The two curves intersect nine times, which is the most that two cubic curves can intersect. [2] We already implicitized the first curve (in section 20.4), so our intersection problem requires us to make the substitutions $x = \frac{-52t_2^3 + 63t_2^2 - 15t_2 + 7}{-37t_2^2 + 3t_2 + 1}$ and $y = \frac{4}{-37t_2^2 + 3t_2 + 1}$ into the implicit equation of curve 1:

$$f(x,y) = -156195x^3 + 60426x^2y - 7056xy^2 + 224y^3 +$$
$$2188998x^2 - 562500xy + 33168y^2 - 10175796x + 1322088y +$$
$$15631624.$$

After multiplying through by $(-37t_2^2 + 3t_2 + 1)^3$, we arrive at the intersection equation:

$$984100t_2^9 - 458200t_2^8 + 8868537t_2^7 - 9420593t_2^6 + 5949408t_2^5$$

[2] Bezout's theorem states that two curves of degree m and n respectively, intersect in mn points, if we include complex points, points at infinity, and multiple intersections.

$$- \ 2282850t_2^4 \ + \ 522890t_2^3 \ - \ 67572t_2^2 \ + \ 4401t_2 \ - \ 109 \ = \ 0.$$

Again, we have carried out this process in exact integer arithmetic to emphasize that this equation is an *exact* representation of the intersection points.

We now compute the roots of this degree 9 polynomial. Those roots are the parameter values of the points of intersection. The (x, y) coordinates of those intersection points can be easily found from the parametric equation of the second curve, and the parameter values on the first curve for the intersection points can be found from the inversion equations. The results are tabulated below.

Intersection Number	t_1 **Parameter of Curve 1**	**Coordinates of Point**	t_2 **Parameter of Curve 2**
1	0.0621	(4.2982,2.3787)	0.3489
2	0.1098	(4.4556,2.9718)	0.1330
3	0.1785	(4.6190,3.4127)	0.9389
4	0.3397	(4.9113,3.2894)	0.9219
5	0.4212	(4.9312,3.2186)	0.0889
6	0.6838	(5.1737,2.2902)	0.5339
7	0.8610	(5.4676,2.3212)	0.5944
8	0.9342	(5.6883,2.8773)	0.8463
9	0.9823	(5.9010,3.6148)	0.0369

The most common curve intersection algorithms are currently based on subdivision. Tests indicate that this implicitization algorithm is faster than subdivision methods for curves of degree two and three, and subdivision methods are faster for curves of degree five and greater [Sederberg et al '86].

8.8 Surfaces

Implicitization and inversion algorithms exist for surfaces, also (see [Sederberg '83] or [Sederberg et al. '84b]). But, whereas curve implicitization yields implicit equations of the same degree as the parametric equations, surface implicitization experiences a degree explosion. A triangular surface patch, whose parametric equations are of the form

$$x \ = \ \frac{\sum_{i+j\le n} x_{ij}s^it^j}{\sum_{i+j\le n} w_{ij}s^it^j} \qquad y \ = \ \frac{\sum_{i+j\le n} y_{ij}s^it^j}{\sum_{i+j\le n} w_{ij}s^it^j}$$

$$z \ = \ \frac{\sum_{i+j\le n} z_{ij}s^it^j}{\sum_{i+j\le n} w_{ij}s^it^j} \qquad i, j \ge 0,$$

generally has an implicit equation of degree n^2. A tensor product surface patch, whose parametric equations are of the form

$$x \ = \ \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} x_{ij}s^it^j}{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij}s^it^j} \qquad y \ = \ \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} y_{ij}s^it^j}{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij}s^it^j}$$

$$z \ = \ \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} z_{ij}s^it^j}{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij}s^it^j},$$

generally has an implicit equation of degree 2mn. Thus, a bicubic patch generally has an implicit equation $f(x, y, z) = 0$ of degree 18. Such an equation has 1330 terms!

Algebraic geometry shares important information on the nature of intersections of parametric surfaces. Recall that Bezout's theorem states that two surfaces of degree m and n respectively intersect in a curve of degree mn. Thus, two bicubic patches generally intersect in a curve of degree 324.

We have noted that bilinear patches have an implicit equation of degree 2; quadratic patches have an implicit equation of degree 4; biquadratic patches have an implicit equation of degree 8, etc. It seems highly curious that there are gaps in this sequence of degrees. Are there no parametric surfaces whose implicit equation is degree 3 or 5 for example? It turns out that parametric surfaces of degree n only *generally* have implicit equations of degree n^2 and that under certain conditions that degree will decrease. To understand the nature of those conditions, we must understand why the implicit equation of a parametric surface is generally n^2. The *degree* of a surface can be thought of either as the degree of its implicit equation, or as the number of times it is intersected by a line. Thus, the degree of the implicit equation of a parametric surface can be found by determining the number of times it is intersected by a line. Consider a parametric surface given by

$$x = \frac{x(s,t)}{w(s,t)} \quad y = \frac{y(s,t)}{w(s,t)} \quad z = \frac{z(s,t)}{w(s,t)},$$

where the polynomials are of degree n. One way we can compute the points at which a line intersects the surface is by intersecting the surface with two planes which contain the line. If one plane is $Ax + By + Cz + Dw = 0$, its intersection with the surface is a curve of degree n in s, t space: $Ax(s,t) + By(s,t) + Cz(s,t) + Dw(s,t) = 0$. The second plane will also intersect the surface in a degree n curve in parameter space. The points at which these two section curves intersect will be the points at which the line intersects the surface. According the Bezout's theorem, two curves of degree n intersect in n^2 points. Thus, the surface is generally of degree n^2.

8.9 Base Points

It may happen that there are values of s, t for which $x(s,t) = y(s,t) = z(s,t) = w(s,t) = 0$. These are known as *base points* in contemporary algebraic geometry. If a base point exists, any plane section curve will contain it. Therefore, it will belong to the set of intersection points of any pair of section curves. However, since a base point maps to something that is undefined in x, y, z space, it does not represent a point at which the straight line intersects the surface, and thus *the existence of a base point diminishes the degree of the implicit equation by one.* Thus, if there happen to be r base points on a degree n parametric surface, the degree of its implicit equation is $n^2 - r$.

For example, it is well known that any quadric surface can be expressed in terms of degree 2 parametric equations. However, in general, a degree 2 parametric surface has an implicit equation of degree 4, and we conclude that there must be two base points. Consider the parametric equations of a sphere of radius r centered at the origin:

$$x = 2rs^u$$
$$y = 2rtu$$
$$z = r(u^2 - s^2 - t^2)$$
$$w = s^2 + t^2 + u^2$$

Homogeneous parameters s, t, u are used to enable us to verify the existence of the two base points: $s = 1,\ t = i,\ u = 0$ and $s = 1,\ t = -i,\ u = 0$.

8.10 Implicitization in Bézier Form

From the paper [47], a Bézier curve can be implicitized as follows. (Note that the value l_{ij} in these notes is equivalent to the value $L_{j,k+1}$ in the paper).

A degree 2 Bézier curve can be implicitized:

$$f(x,y) = \begin{vmatrix} l_{01}(x,y) & l_{02}(x,y) \\ l_{02}(x,y) & l_{12}(x,y) \end{vmatrix}$$

and a degree 3 Bézier curve can be implicitized

$$f(x,y) = \begin{vmatrix} l_{01}(x,y) & l_{02}(x,y) & l_{03}(x,y) \\ l_{02}(x,y) & l_{03}(x,y) + l_{12}(x,y) & l_{13}(x,y) \\ l_{03}(x,y) & l_{13}(x,y) & l_{23}(x,y) \end{vmatrix}$$

where

$$l_{ij}(x,y) = \binom{n}{i}\binom{n}{j} w_i w_j \begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}$$

with n the degree of the curve, and x_i, y_i, w_i the coordinates and weight of the i^{th} control point.

For a general degree n curve, the implicit equation is

$$f(x,y) = \begin{vmatrix} L_{0,0}(x,y) & \cdot\ \cdot\ \cdot & L_{0,n-1}(x,y) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ L_{n-1,0}(x,y) & \cdot\ \cdot\ \cdot & L_{n-1,n-1}(x,y) \end{vmatrix}$$

where

$$L_{i,j} = \sum_{\substack{k \le \min(i,j) \\ k+m=i+j+1}} l_{km}.$$

Inversion is accomplished by solving the set of equations

$$\begin{bmatrix} l_{01}(x,y) & l_{02}(x,y) & l_{03}(x,y) \\ l_{02}(x,y) & l_{03}(x,y) + l_{12}(x,y) & l_{13}(x,y) \\ l_{03}(x,y) & l_{13}(x,y) & l_{23}(x,y) \end{bmatrix} \begin{Bmatrix} (1-t)^2 \\ t(1-t) \\ t^2 \end{Bmatrix} = 0$$

for the degree three case, and

$$\begin{bmatrix} l_{01}(x,y) & l_{02}(x,y) \\ l_{02}(x,y) & l_{12}(x,y) \end{bmatrix} \begin{Bmatrix} (1-t) \\ t \end{Bmatrix} = 0$$

for degree two.