150 points
Wednessday, 19 April 2000, 11a.m.—2 p.m.

Three pages of notes allowed. Show all your work.

1. (20 points) The following questions deal with the execution time of algorithms and are multiple choice. If for $n = 10$, the algorithm takes k units of time to execute, approximately how long would the algorithm take if $n = 20$? Possible answers are:

a. k b. $2k$ c. $4k$ d. $8k$. e. $\geq 16k$

____ **a.** The de Casteljau algorithm for a degree n Bézier curve.

____ **b.** The algorithm for converting the parametric equations of a degree n Bézier curve to power basis.

____ **c.** Given $n+1$ points $\mathbf{P}(0)$, $\mathbf{P}(\delta)$, $\mathbf{P}(2\delta)$, $\ldots$, $\mathbf{P}(n\delta)$, on a curve known to be a degree n polynomial Bézier curve (but you don't know the control points) find $\mathbf{P}((n+1)\delta)$ using forward differencing.

____ **d.** Compute the tangent vector of a degree n rational Bézier curve at $t = 0$.

____ **e.** Neville's algorithm for a degree n curve.

____ **f.** Evaluate a degree n Newton polynomial (you already know the coefficients of the polynomial).

____ **g.** Degree elevate a degree n Bézier curve.

____ **h.** An algorithm to find the minimum number of line segments (with evenly spaced parameter values) needed to plot a degree n Bézier curve so that the error is less than a specified value.

____ **i.** Subdivide a degree $n \times n$ tensor-product surface patch.

____ **j.** Compute the curvature at $t = 1$ for a degree n rational Bézier curve.

2. (8 points) A cubic polynomial $f(t)$ satisfies

$$f(1) = 1; \qquad f(2) = 3; \qquad f(5) = 2; \qquad f(7) = 1.$$

Find $f(3)$ using Newton polynomials.

3. (10 points) A cubic curve $\mathbf{P}(t)$ interpolates four points as follows:

$$\mathbf{P}(1) = (1,1); \qquad \mathbf{P}(2) = (2,3); \qquad \mathbf{P}(4) = (4,2); \qquad \mathbf{P}(5) = (5,1);$$

Use Neville's scheme to find $\mathbf{P}(3)$.

4. (8 points) A degree four polynomial Bézier curve has control points

$$\mathbf{P}_0 = (0,0), \quad \mathbf{P}_1 = (1,6), \quad \mathbf{P}_2 = (2,8), \quad \mathbf{P}_3 = (3,6), \quad \mathbf{P}_4 = (4,0).$$

Use the error bounds formula for Lagrange interpolation to find an upper bound on the distance from the curve to the x axis.

5. (5 points) For the curve in problem 4, find an upper bound on the distance from the curve to the x axis using the convex hull property.

6. (5 points) Use Euclid's algorithm to find the GCD of 5292 and 2860.

7. (10 points) Find the implicit equation of the degree 2 Bézier curve with control points

$$\mathbf{P}_0 = (0,0) \qquad \mathbf{P}_1 = (2,0); \quad \mathbf{P}_2 = (4,4).$$

8. (10 points) Find an inversion equation for the curve

$$x = t^2 + 2t + 1; \qquad y = 2t^2 + t - 2.$$

Use your inversion equation to compute the parameter for the point $(0, -1)$.

9. (10 points) Analyze the error that can occur in the forward difference method as follows. Define

$$f(0) = [.999, 1.001] \quad f(1) = [1.999, 2.001] \quad f(2) = [3.999, 4.001]$$

where f(t) is a degree two polynomial. In words, $f(0)$ can be any number in the interval $[.999, 1.001]$ etc. Using forward differencing, compute $f(3)$, $f(4)$, and $f(5)$. That is, give the interval that contains all possible values of $f(3)$. (Hint: Make the fourth row in the difference table to be all zeros, not intervals.)

Answer: $f(3) =$____________ $f(4) =$____________ $f(5) =$____________

10. (6 points) In the preceeding problem, let $w(3)$ be the width of the interval of $f(3)$, $w(4)$ be the width of $f(4)$ and $w(5)$ be the width of $f(5)$. (The width of an interval $[a, b]$ is $b - a$.) If we know that $w(t)$ is a degree two polynomial, compute $w(100)$ using Lagrange polynomials. Use only $w(3)$, $w(4)$ and $w(5)$ in the Lagrange polynomial, not $w(0)$, $w(1)$ or $w(2)$.

11. (8 points) Find the vector that is normal to the bi-quadratic surface patch with control points:

$$\mathbf{P}_{00} = (0,0,0) \quad \mathbf{P}_{10} = (4,8,0) \quad \mathbf{P}_{20} = (8,0,0)$$

$$\mathbf{P}_{01} = (0,0,4) \quad \mathbf{P}_{11} = (4,16,4) \quad \mathbf{P}_{21} = (8,0,4)$$

$$\mathbf{P}_{02} = (0,0,8) \quad \mathbf{P}_{12} = (4,8,8) \quad \mathbf{P}_{22} = (8,0,8)$$

at $s = \frac{1}{2} \quad t = 0$.

12. (10 points) An explicit rational curve is given by the power-basis equations:

$$x = t = t\frac{3t^2 + 6t - 6}{3t^2 + 6t - 6} = \frac{3t^3 + 6t^2 - 6t}{0t^3 + 3t^2 + 6t - 6}$$

$$y = \frac{-12t^3 + 18t^2 - 3t + 6}{0t^3 + 3t^2 + 6t - 6}$$

What are the control points and weights if this curve is converted to rational Bézier form?

13. (10 points) A quadratic Bézier curve has control points

$$\mathbf{P}_0 = (0,0) \quad \mathbf{P}_1 = (-10,10) \quad \mathbf{P}_2 = (40,100).$$

Find the minimum number of line segments (of equal parameter interval) needed to approximate this curve to within a tolerance d. That is, the distance between any line segment and the curve will not exceed d.

14. (14 points) Circle T (for TRUE) or F (for FALSE) for each statement:

T F A curve which always obeys the convex hull property is coordinate system independent.

T F A curve which is variation diminishing obeys the convex hull property.

T F The Lagrange basis obeys the convex hull property.

T F If the polynomial $f(t)$ has a double root, the resultant of $f(t)$ and $f'(t)$ is *always* zero.

T F Any cubic B-spline that has a knot vector [0,1,2,3,4,4,5,6,7,8] is exactly C^1 at $t = 4$.

T F The genus of any degree n Bézier curve is $(n+1)$.

T F Any Bézier curve can be represented as a single B-spline curve.

15. (8 points) The equation of a degree 2 planar rational curve

$$x = \frac{t^2-1}{t^2+1}; \qquad y = \frac{2t}{t^2+1}$$

is substituted into the implicit equation of a second degree 2 curve $f(x,y) = 0$ yielding

$$t^3 - 2t^2 = 0.$$

What are the (x, y) coordinates of the points at which the two curves intersect?

16. (8 points) For this periodic B-spline, insert a knot at t=6. Give the coordinates of the control points and the new knot vector.